

# A SPECTRAL MAPPING THEOREM FOR PERTURBED ORNSTEIN-UHLENBECK OPERATORS ON $L^2(\mathbb{R}^d)$

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ABSTRACT. We consider Ornstein-Uhlenbeck operators on  $L^2(\mathbb{R}^d)$  perturbed by a radial potential  $V$ . Under weak assumptions on  $V$  we prove a spectral mapping theorem for the generated semigroup. The proof relies on a perturbative construction of the resolvent, based on angular separation, and the Gearhart-Prüß Theorem.

## 1. INTRODUCTION

We consider a class of operators generated by the formal differential expression

$$\mathcal{L}_V u(x) := \Delta u(x) - 2x \cdot \nabla u(x) + V(|x|)u(x) \quad (1.1)$$

for  $u : \mathbb{R}^d \rightarrow \mathbb{C}$  with a complex-valued radial potential  $V : [0, \infty) \rightarrow \mathbb{C}$ . The study of elliptic and parabolic problems with unbounded coefficients is motivated by many applications in science, engineering, and economics. Operators as in (1.1) are prototypes of this kind and attracted a lot of interest in the mathematical literature. We refer to the monograph [16] for recent developments in this field.

A natural space for the analysis of  $\mathcal{L}_V$  is the weighted  $L_w^2(\mathbb{R}^d)$  with the Gaussian weight  $w(x) = e^{-|x|^2}$ . The reason for this is that the *free operator*  $\mathcal{L}_0$  is symmetric on  $L_w^2(\mathbb{R}^d)$ . However, in this paper we consider  $\mathcal{L}_V$  on  $L^2(\mathbb{R}^d)$  without weight, which is motivated in the following.

With a suitable domain (see below) the formal expression  $\mathcal{L}_V$  has a realization as an unbounded operator  $L$  on  $L^2(\mathbb{R}^d)$  which generates a strongly continuous one-parameter semigroup  $\{S(t) : t \geq 0\}$ . This shows that the  $L^2$ -setting without weight is very natural, too. The operator  $L$  is highly non-self-adjoint and the complex half-plane  $\{z \in \mathbb{C} : \operatorname{Re} z \leq d\}$  is contained in its spectrum. Thus,  $L$  has in some

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sense the worst possible spectral structure that still allows for the generation of a semigroup. This makes the analysis of  $S(t)$  mathematically interesting and challenging since the application of general “soft” arguments is largely precluded. Furthermore, besides the well-known applications to probability and mathematical finance, operators of the form (1.1) occur very naturally in the study of self-similar solutions to nonlinear parabolic equations. To see this, consider for instance the equation

$$\partial_t u(t, x) = \Delta_x u(t, x) + F(u(t, x), |x|)$$

where  $F$  is some given nonlinearity that allows for the existence of a radial self-similar solution of the form  $u_0(t, x) = (1 - t)^{-\beta} f(\frac{|x|}{\sqrt{1-t}})$ . In order to analyse the stability of  $u_0$ , it is standard to introduce *similarity coordinates*  $\tau = -\log(1 - t)$ ,  $\xi = \frac{x}{\sqrt{1-t}}$ . If  $F$  scales suitably, the change of variables  $(t, x) \mapsto (\tau, \xi)$  leads to an equation of the form

$$\begin{aligned} \partial_\tau \tilde{u}(\tau, \xi) = & \Delta_\xi \tilde{u}(\tau, \xi) - \frac{1}{2} \xi \cdot \nabla_\xi \tilde{u}(\tau, \xi) - \beta \tilde{u}(\tau, \xi) \\ & + \partial_1 F(f(|\xi|), |\xi|) \tilde{u}(\tau, \xi) + \text{nonlinear terms} \end{aligned}$$

for  $\tilde{u}(\tau, \xi) = (1 - t)^\beta [u(t, x) - u_0(t, x)]$ . Consequently, the linear part on the right-hand side is an Ornstein-Uhlenbeck operator as in (1.1). In such a situation one is naturally led to the unweighted setting since Sobolev spaces with *decaying* Gaussian weights are in general not suitable to study nonlinear problems.

As usual, the important question for applications is whether one can derive growth estimates for the semigroup  $S(t)$  by merely looking at the spectrum of  $L$ , which is typically the only accessible information. In the present paper we answer this question in the affirmative for the class of operators defined by Eq. (1.1). We prove the strongest possible result in this context, namely that the spectrum of  $S(t)$  is completely determined by the spectrum of  $L$ .

**Theorem 1.1** (Spectral mapping for Ornstein-Uhlenbeck operators). *Suppose  $V : [0, \infty) \rightarrow \mathbb{C}$  satisfies  $|V(r)| \leq C\langle r \rangle^{-2}$ ,  $|V'(r)| \leq C\langle r \rangle^{-3}$  for all  $r \geq 0$  and some constant  $C > 0$ . Set*

$$\mathcal{D}(\tilde{L}) := \{u \in H^2(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d) : \mathcal{L}_V u \in L^2(\mathbb{R}^d)\},$$

*and define  $\tilde{L} : \mathcal{D}(\tilde{L}) \subset L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  by  $\tilde{L}u := \mathcal{L}_V u$ . Then  $\tilde{L}$  is densely defined, closable, and its closure  $L$  generates a strongly continuous one-parameter semigroup  $\{S(t) : t \geq 0\}$  of bounded operators on  $L^2(\mathbb{R}^d)$  such that the spectral mapping*

$$\sigma(S(t)) \setminus \{0\} = \{e^{t\lambda} : \lambda \in \sigma(L)\}$$

*holds for all  $t \geq 0$ .*

**1.1. Remarks.** We remark that the proof of Theorem 1.1 considerably simplifies if  $V \leq 0$  because in this case one has  $\sigma(L) = \sigma(L_0)$  and from [18] it follows that  $\sigma(L_0) = \{z \in \mathbb{C} : \operatorname{Re} z \leq d\}$ , see also Lemma 2.2 below. However, as explained in the introduction, our main motivation for studying this problem comes from the stability of self-similar solutions. In this context there is no reason to believe that the potential has a sign. In fact, the most interesting situations occur if there exist self-similar solutions with a finite number of unstable modes. The corresponding potentials will then have finitely many zeros. Since our proof works equally well for complex-valued potentials, we decided to state Theorem 1.1 in this generality.

Determining the spectrum of  $L$  is a different problem which we do not touch upon in this paper. After all, the spectrum of  $L$  depends on the concrete form of  $V$ . On the other hand, as a by-product of our investigations, we can at least deduce the following nontrivial property.

**Theorem 1.2.** *Let  $\sigma'(L) := \sigma(L) \setminus \sigma(L_0)$ . Then, for any  $b > 0$ , the set*

$$\sigma'(L) \cap \{d + b + i\omega : \omega \in \mathbb{R}\}$$

*is bounded.*

In this respect we also remark that our construction seems to imply that  $\sigma'(L)$  is discrete since the addition of the potential  $V$  does not change the asymptotics of the involved ODEs. However, we do not elaborate on this any further since in typical applications the potential  $V$  induces a relatively compact operator and the abstract theory implies that  $\sigma'(L)$  consists of eigenvalues only. As a matter of fact, relative compactness can already be deduced from very mild decay properties of the potential, see e.g. [9].

Finally, we would like to mention that it is possible to weaken the assumptions on  $V$  considerably. For instance, inspection of the proof shows that  $V \in W^{1,1}(\mathbb{R}_+)$  suffices for the argument to go through. However, for the sake of simplicity we do not prove Theorem 1.1 in this generality.

**1.2. Further discussion and related work.** Ornstein-Uhlenbeck operators are mostly studied in suitable weighted spaces with invariant measures, e.g. [17, 20, 21, 14, 6, 16]. However, there is also a growing interest in the corresponding operators acting on spaces with more general weights [28] or on unweighted  $L^p$ -spaces as in the present paper, see e.g. [18, 1, 12, 19].

In general, the question of spectral mapping between the semigroup and its generator is of uttermost importance for applications since typically, the only way to determine the stability of a time-evolution system is to study the spectrum of its generator. Unfortunately, spectral mapping is not stable under bounded perturbations. If the perturbed semigroup does not have “nice” properties such as eventual norm continuity, it can be very difficult to prove a suitable spectral mapping theorem. Parabolic equations in non-self-adjoint settings are a prominent example where spectral mapping is nontrivial, see e.g. [11] and references therein. Furthermore, hyperbolic equations are an important class of evolution problems where difficult problems related to spectral mapping occur since the spectrum contains the imaginary axis. Although there are many positive results, see e.g. [8] for the case of Schrödinger equations and [15, 2] for hyperbolic systems, it was precisely in this context where Renardy constructed his by now famous counterexample [25]. It shows that very natural, relatively compact perturbations of the wave equation can destroy spectral mapping, even in a standard  $L^2$ -space. This simple example came as a shock although many counterexamples to spectral mapping were known at the time, e.g. [13, 5, 29, 10]. However, there was a widespread belief that such a pathology is confined to rather artificially constructed situations that never occur in real-world applications. In view of this, Renardy’s example is very disturbing. On the other hand, it is known that in Hilbert spaces “most” bounded perturbations preserve spectral mapping [26]. This is a positive result from a psychological point of view but it cannot be used to deduce spectral mapping for a concrete problem.

To the knowledge of the authors, the most general “abstract” conditions that guarantee spectral mapping are based on norm continuity properties, see [3]. However, we do not see how to apply the theory from [3] to the problem at hand, see Appendix A for a discussion on this. That is why we choose a more explicit approach. The key tool is the Gearhart-Prüß Theorem [7, 24] which reduces the question of spectral mapping to uniform bounds on the resolvent with respect to the imaginary part of the spectral parameter. Consequently, we perform an explicit perturbative construction of the resolvent along vertical lines in the complex plane, for large imaginary parts of the spectral parameter. This is possible because the potential is assumed to be radial which allows us to reduce the spectral problem to an infinite number of decoupled ODEs, one for each value of the angular momentum parameter  $\ell$ . For each fixed  $\ell$  we construct the “reduced” resolvent by asymptotic ODE methods based on the Liouville-Green transform, and we establish  $L^2$ -bounds that hold uniformly in  $\ell$ . These allow us to obtain the

desired bounds for the “full” resolvent by summing over  $\ell$  and spectral mapping follows from the Gearhart-Prüß Theorem.

**1.3. Notation.** We use standard Lebesgue and Sobolev spaces denoted by  $L^p$ ,  $W^{k,p}$ ,  $H^k = W^{k,2}$ , and  $\mathcal{S}$  is the Schwartz space. Furthermore, for  $f : \mathbb{R}_+ \rightarrow \mathbb{C}$  we write

$$\|f\|_{L^2_{\text{rad}}(\mathbb{R}^d)}^2 := \int_0^\infty |f(r)|^2 r^{d-1} dr$$

where  $\mathbb{R}_+ := [0, \infty)$ . The letter  $C$  (possibly with indices to indicate dependencies) denotes a positive constant that might change its value at each occurrence. We write  $f(r) = O_{\mathbb{C}}(g(r))$  if  $|f(r)| \leq C|g(r)|$  and  $|f'(r)| \leq C|g'(r)|$  (the subscript  $\mathbb{C}$  indicates that  $f$  might be complex-valued). As usual,  $A \lesssim B$  means  $A \leq CB$  where  $C$  is independent of all the parameters that occur in the inequality. We also write  $A \simeq B$  if  $A \lesssim B$  and  $B \lesssim A$ .  $A \gg B$  means that  $A \geq CB$  for  $C$  sufficiently large. Furthermore, we frequently use the “japanese bracket” notation  $\langle x \rangle := \sqrt{1 + |x|^2}$ . For the Wronskian  $W(f, g)$  of two functions  $f$  and  $g$  we use the convention  $W(f, g) = fg' - f'g$ . The domain of a closed operator  $A$  on a Banach space is denoted by  $\mathcal{D}(A)$  and we write  $\sigma(A)$  for its spectrum and  $\rho(A) = \mathbb{C} \setminus \sigma(A)$  for its resolvent set. Finally, in the technical part we restrict ourselves to  $d \geq 3$  in order to avoid technicalities involving logarithmic corrections. With minor modifications, the same construction can be performed in the case  $d = 2$ .

## 2. PRELIMINARIES

**2.1. Generation of the semigroup.** For the sake of completeness we include the generation result which is of course well known [16].

**Lemma 2.1.** *Define  $\tilde{L}_0 : \mathcal{D}(\tilde{L}) \subset L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  by  $\tilde{L}_0 u := \tilde{L}u - Vu$  where  $\tilde{L}$  and  $V$  are from Theorem 1.1. Then the operator  $\tilde{L}_0$  is densely defined, closable, and its closure  $L_0$  generates a strongly continuous one-parameter semigroup  $\{S_0(t) : t \geq 0\}$  which satisfies*

$$\|S_0(t)\|_{L^2(\mathbb{R}^d)} \leq e^{dt}$$

for all  $t \geq 0$ . As a consequence,  $L$  generates a semigroup  $\{S(t) : t \geq 0\}$  satisfying

$$\|S(t)\|_{L^2(\mathbb{R}^d)} \leq e^{Mt}$$

for all  $t \geq 0$  where  $M = d + \|V\|_{L^\infty(\mathbb{R}_+)}$ .

*Proof.* Since  $C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(\tilde{L})$ , it is obvious that  $\tilde{L}_0$  is densely defined and integration by parts yields

$$\operatorname{Re}(\tilde{L}_0 u | u)_{L^2(\mathbb{R}^d)} = - \int_{\mathbb{R}^d} |\nabla u(x)|^2 dx + d \int_{\mathbb{R}^d} |u(x)|^2 dx \leq d \|u\|_{L^2(\mathbb{R}^d)}^2$$

for all  $u \in \mathcal{D}(\tilde{L})$  (the boundary terms vanish by the density of  $C_c^\infty(\mathbb{R}^d)$  in  $H^2(\mathbb{R}^d)$ ). Now we claim that the operator  $2d - \tilde{L}_0$  has dense range. To prove this, it suffices to show that the equation  $(2d - \tilde{L}_0)u = f$  has a solution  $u \in H^2(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$  for any given  $f \in \mathcal{S}(\mathbb{R}^d)$ . On the Fourier side this equation reads

$$(|\xi|^2 - 2\xi \cdot \nabla) \hat{u}(\xi) = \hat{f}(\xi)$$

which is solved by

$$\hat{u}(r\omega) = \int_r^\infty \frac{e^{-\frac{1}{4}(s^2-r^2)}}{2s} \hat{f}(s\omega) ds$$

where we introduced polar coordinates  $r = |\xi|$  and  $\omega = \xi/|\xi|$ . We obtain

$$\hat{u}(r\omega) r^{\frac{d-1}{2}} = \int_0^\infty K(r, s) \hat{f}(s\omega) s^{\frac{d-1}{2}} ds$$

with  $K(r, s) = \frac{1}{2} r^{\frac{d-1}{2}} e^{-\frac{1}{4}(s^2-r^2)} s^{-\frac{d+1}{2}} 1_{[0, \infty)}(s-r)$ . The kernel  $K$  satisfies the bound  $|K(r, s)| \lesssim \min\{r^{-1}, s^{-1}\}$  and hence, it induces a bounded operator on  $L^2(\mathbb{R}_+)$  (see e.g. [4], Lemma 5.5). Consequently, we infer

$$\begin{aligned} \|\hat{u}\|_{L^2(\mathbb{R}^d)}^2 &= \int_{\mathbb{S}^{d-1}} \|\hat{u}(\cdot\omega)\|_{L^2_{\text{rad}}(\mathbb{R}^d)}^2 d\sigma(\omega) \lesssim \int_{\mathbb{S}^{d-1}} \|\hat{f}(\cdot\omega)\|_{L^2_{\text{rad}}(\mathbb{R}^d)}^2 d\sigma(\omega) \\ &= \|\hat{f}\|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

Furthermore, on the support of the kernel  $K(r, s)$  we have  $r \leq s$  and thus, by the same reasoning as above we obtain the bound

$$\|\langle \cdot \rangle^N \hat{u}\|_{L^2(\mathbb{R}^d)} \leq C_N \|\langle \cdot \rangle^N \hat{f}\|_{L^2(\mathbb{R}^d)}$$

for any  $N \in \mathbb{N}$  and the right-hand side is finite since  $\hat{f}$  is Schwartz. By taking the inverse Fourier transform of  $\hat{u}$ , we obtain a function  $u \in H^2(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$  which satisfies  $(2d - \tilde{L}_0)u = f$ . Consequently, the Lumer-Phillips Theorem yields the existence of the semigroup  $\{S_0(t) : t \geq 0\}$  with the stated bound. The statements for  $L$  and  $S(t)$  follow from the Bounded Perturbation Theorem.  $\square$

We also recall the spectral structure of the free Ornstein-Uhlenbeck operator  $L_0$ . Note that the following lemma shows in particular that the growth bound  $\|S_0(t)\|_{L^2(\mathbb{R}^d)} \leq e^{dt}$  is sharp and that  $S$  is a general

$C_0$ -semigroup without additional properties (such as eventual norm-continuity).

**Lemma 2.2.** *The spectrum of  $L_0$  (defined in Lemma 2.1) is given by*

$$\sigma(L_0) = \{z \in \mathbb{C} : \operatorname{Re} z \leq d\}.$$

Furthermore, we have  $\sigma(L_0) \subset \sigma(L)$ .

*Proof.* The statement on  $\sigma(L_0)$  follows from [18] and  $\sigma(L_0) \subset \sigma(L)$  is proved in Appendix B.  $\square$

**2.2. Angular decomposition of the resolvent.** From now on we set  $R(\lambda) := (\lambda - L)^{-1}$  for  $\lambda \in \rho(L)$  where  $L$  is from Theorem 1.1. We exploit the assumed radial symmetry of the potential  $V$  by an angular decomposition. Let  $Y_{\ell,m} : \mathbb{S}^{d-1} \rightarrow \mathbb{C}$  denote a standard spherical harmonic, i.e., an  $L^2(\mathbb{S}^{d-1})$ -normalized eigenfunction of the Laplace-Beltrami operator on  $\mathbb{S}^{d-1}$  with eigenvalue  $\ell(\ell + d - 2)$ . We denote by  $\Omega_d \subset \mathbb{N}_0 \times \mathbb{Z}$  the set of admissible values of  $(\ell, m)$ . The precise domain of  $m$  is irrelevant for us but we note that all  $\ell \in \mathbb{N}_0$  occur. For  $f \in L^2(\mathbb{R}^d)$  we define

$$[P_{\ell,m}f](r) := \int_{\mathbb{S}^{d-1}} f(r\omega) \overline{Y_{\ell,m}(\omega)} d\sigma(\omega)$$

and Cauchy-Schwarz implies that  $P_{\ell,m}$  is a bounded operator from  $L^2(\mathbb{R}^d)$  to  $L^2_{\text{rad}}(\mathbb{R}^d)$  with operator norm 1. Furthermore, we define a bounded operator  $Q_{\ell,m} : L^2_{\text{rad}}(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  by

$$[Q_{\ell,m}g](x) := g(|x|) Y_{\ell,m}\left(\frac{x}{|x|}\right)$$

and again, the operator norm of  $Q_{\ell,m}$  is 1. We also note that  $Q_{\ell,m}P_{\ell,m} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  is a projection. The fact that the formal differential operator  $\mathcal{L}_V$  in polar coordinates separates into a radial and an angular component can now be phrased in operator language as follows.

**Lemma 2.3.** *Let  $\lambda \in \rho(L)$ . Then  $[R(\lambda), Q_{\ell,m}P_{\ell,m}] = 0$  for all  $(\ell, m) \in \Omega_d$ .*

*Proof.* It suffices to prove the identity  $R(\lambda)Q_{\ell,m}P_{\ell,m} = Q_{\ell,m}P_{\ell,m}R(\lambda)$  on a dense subset of  $L^2(\mathbb{R}^d)$ . So let  $f \in C_c^\infty(\mathbb{R}^d)$ . By Lemma 2.2 we have  $\lambda \in \rho(L_0)$  and we set  $u = R_0(\lambda)f$  where  $R_0(\lambda) := (\lambda - L_0)^{-1}$  is the resolvent of the free Ornstein-Uhlenbeck operator  $L_0$ . By elliptic regularity we have  $u \in C^\infty(\mathbb{R}^d)$  and we infer

$$[P_{\ell,m}L_0u](r) = \int_{\mathbb{S}^{d-1}} (D_r + \frac{1}{r^2}\Delta_\omega)u(r\omega) \overline{Y_{\ell,m}(\omega)} d\sigma(\omega), \quad r > 0$$

where  $D_r = \partial_r^2 + \frac{d-1}{r}\partial_r - 2r\partial_r$  and  $-\Delta_\omega$  is the Laplace-Beltrami operator on  $\mathbb{S}^{d-1}$ . By dominated convergence and the fact that  $\Delta_\omega$  is symmetric on  $L^2(\mathbb{S}^{d-1})$  we infer

$$\begin{aligned} [P_{\ell,m}L_0u](r) &= \left[ D_r - \frac{\ell(\ell+d-2)}{r^2} \right] \int_{\mathbb{S}^{d-1}} u(r\omega) \overline{Y_{\ell,m}(\omega)} d\sigma(\omega) \\ &= \left[ D_r - \frac{\ell(\ell+d-2)}{r^2} \right] P_{\ell,m}u(r) \end{aligned}$$

and this implies

$$Q_{\ell,m}P_{\ell,m}(\lambda - L_0)u = (\lambda - L_0)Q_{\ell,m}P_{\ell,m}u.$$

Consequently, we obtain  $R_0(\lambda)Q_{\ell,m}P_{\ell,m}f = Q_{\ell,m}P_{\ell,m}R_0(\lambda)f$ . The claimed  $[R(\lambda), Q_{\ell,m}P_{\ell,m}] = 0$  follows now from the identity

$$R(\lambda) = R_0(\lambda)[1 - VR_0(\lambda)]^{-1}$$

and the fact that  $V$  is radial (invertibility of  $1 - VR_0(\lambda)$  is a consequence of  $\lambda - L = [1 - VR_0(\lambda)](\lambda - L_0)$  and  $\lambda \in \rho(L_0) \cap \rho(L)$ ).  $\square$

**Definition 2.4.** For  $\lambda \in \rho(L)$  and  $(\ell, m) \in \Omega_d$  we define the *reduced resolvent*  $R_{\ell,m}(\lambda) : L_{\text{rad}}^2(\mathbb{R}^d) \rightarrow L_{\text{rad}}^2(\mathbb{R}^d)$  by

$$R_{\ell,m}(\lambda) := P_{\ell,m}R(\lambda)Q_{\ell,m}.$$

**Lemma 2.5.** *For every  $\lambda \in \rho(L)$  we have the bound*

$$\|R(\lambda)\|_{L^2(\mathbb{R}^d)} \leq \sup_{(\ell,m) \in \Omega_d} \|R_{\ell,m}(\lambda)\|_{L_{\text{rad}}^2(\mathbb{R}^d)}.$$

*Proof.* For brevity we write  $\sum_{\ell,m} := \sum_{(\ell,m) \in \Omega_d}$ . Every  $f \in L^2(\mathbb{R}^d)$  can be expanded as

$$f = \sum_{\ell,m} Q_{\ell,m}P_{\ell,m}f$$

and the expansion converges in  $L^2(\mathbb{R}^d)$ . Moreover, Parseval's identity and the monotone convergence theorem yield

$$\|f\|_{L^2(\mathbb{R}^d)}^2 = \sum_{\ell,m} \|P_{\ell,m}f\|_{L_{\text{rad}}^2(\mathbb{R}^d)}^2.$$

Consequently, by Lemma 2.3 and  $(Q_{\ell,m}P_{\ell,m})^2 = Q_{\ell,m}P_{\ell,m}$  we infer

$$\begin{aligned} R(\lambda)f &= \sum_{\ell,m} Q_{\ell,m}P_{\ell,m}R(\lambda)f = \sum_{\ell,m} Q_{\ell,m}P_{\ell,m}R(\lambda)Q_{\ell,m}P_{\ell,m}f \\ &= \sum_{\ell,m} Q_{\ell,m}R_{\ell,m}(\lambda)P_{\ell,m}f \end{aligned}$$



which implies

$$\begin{aligned}
\|R(\lambda)f\|_{L^2(\mathbb{R}^d)}^2 &\leq \sum_{\ell,m} \|R_{\ell,m}(\lambda)P_{\ell,m}f\|_{L^2_{\text{rad}}(\mathbb{R}^d)}^2 \\
&\leq \sup_{\ell,m} \|R_{\ell,m}(\lambda)\|_{L^2_{\text{rad}}(\mathbb{R}^d)}^2 \sum_{\ell,m} \|P_{\ell,m}f\|_{L^2_{\text{rad}}(\mathbb{R}^d)}^2 \\
&\leq \sup_{\ell,m} \|R_{\ell,m}(\lambda)\|_{L^2_{\text{rad}}(\mathbb{R}^d)}^2 \|f\|_{L^2(\mathbb{R}^d)}^2.
\end{aligned}$$

□

The main result of the present paper is in fact the following estimate on the reduced resolvent.

**Theorem 2.6.** *Let  $b > 0$ . Then the reduced resolvent  $R_{\ell,m}$  satisfies*

$$\|R_{\ell,m}(d + b + i\omega)\|_{L^2_{\text{rad}}(\mathbb{R}^d)} \leq C$$

for all  $\omega \gg 1$  and all  $(\ell, m) \in \Omega_d$ .

**2.3. Reduction of Theorems 1.1 and 1.2 to Theorem 2.6.** Now we show that Theorem 2.6 implies Theorems 1.1 and 1.2. The rest of the paper is then devoted to the proof of Theorem 2.6.

**Lemma 2.7.** *Theorem 2.6 implies Theorems 1.1 and 1.2.*

*Proof.* First of all we note that by complex conjugation, the stated resolvent bound in Theorem 2.6 holds for large negative  $\omega$  as well. We may assume  $t > 0$  and use the common abbreviation

$$e^{t\sigma(L)} := \{e^{tz} : z \in \sigma(L)\}.$$

Recall that the inclusion  $e^{t\sigma(L)} \subset \sigma(S(t))$  always holds. Thus, in order to show Theorem 1.1, it suffices to prove that  $\mathbb{C}^* \setminus e^{t\sigma(L)} \subset \rho(S(t))$ , where  $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ . Now assume that  $\lambda \in \mathbb{C}^* \setminus e^{t\sigma(L)}$  and suppose  $\lambda = e^{tz}$  for some  $z \in \mathbb{C}$ . Then we must have  $z \in \rho(L)$  since otherwise  $\lambda \in e^{t\sigma(L)}$ . Thus, we obtain

$$\{z \in \mathbb{C} : e^{tz} = \lambda\} \subset \rho(L).$$

Furthermore, Lemma 2.2 shows that  $\frac{1}{t} \log |\lambda| > d$  and by assumption and Lemma 2.5 we have the bound

$$\|R(\frac{1}{t} \log |\lambda| + \frac{i}{t} \arg \lambda + \frac{2\pi i k}{t})\|_{L^2(\mathbb{R}^d)} \leq C$$

for all  $k \in \mathbb{Z}$  (if  $\frac{1}{t} \log |\lambda| > d + \|V\|_{L^\infty(\mathbb{R}_+)}$ , the stated bound follows directly from the growth bound in Lemma 2.1). Consequently, the set

$$\{\|R(z)\|_{L^2(\mathbb{R}^d)} : e^{tz} = \lambda\} \subset \mathbb{R}$$

is bounded and the Gearhart-Prüß Theorem (see [24], Theorem 3) yields  $\lambda \in \rho(S(t))$ . This proves Theorem 1.1.

To prove Theorem 1.2, it suffices to note that, for given  $b > 0$ , it is a consequence of the expansion in the proof of Lemma 2.5 and Theorem 2.6 that the resolvent  $R(d+b+i\omega)$  exists, provided  $|\omega|$  is large enough. This implies Theorem 1.2.  $\square$

### 3. A NONTECHNICAL OUTLINE OF THE RESOLVENT CONSTRUCTION

Since the rest of the paper is very technical, we outline the main steps of our construction in a less formal fashion. This should aid the interested reader when going through the details of the proof.

**3.1. Setup.** Our goal is to construct the reduced resolvent  $R_{\ell,m}(d+b+i\omega)$ , at least for large values of  $\omega$ . If we set  $u_{\ell,m} := R_{\ell,m}(d+b+i\omega)f$ , then  $u_{\ell,m}$  satisfies the ODE

$$\begin{aligned} u''_{\ell,m}(r) + \frac{d-1}{r}u'_{\ell,m}(r) - 2ru'_{\ell,m}(r) - \frac{\ell(\ell+d-2)}{r^2}u_{\ell,m}(r) \\ + V(r)u_{\ell,m}(r) - \lambda u_{\ell,m}(r) = -f_{\ell,m}(r) \end{aligned} \quad (3.1)$$

where  $f_{\ell,m}(r) = (f(r \cdot)|Y_{\ell,m})_{\mathbb{S}^{d-1}}$ . It is convenient to transform Eq. (3.1) to normal form which is achieved by setting  $u_{\ell,m}(r) = r^{-\frac{d-1}{2}}e^{r^2/2}v(r)$ , where we suppress now the subscripts  $\ell, m$  on  $v$  in order to avoid notational clutter. We obtain

$$\begin{aligned} v''(r) - r^2v(r) - \frac{(d+2\ell-1)(d+2\ell-3)}{4r^2}v(r) - (\lambda-d)v(r) \\ + V(r)v(r) = -r^{\frac{d-1}{2}}e^{-r^2/2}f_{\ell,m}(r) \end{aligned} \quad (3.2)$$

and the goal is to invert this operator, i.e., we have to compute  $v$  in terms of  $f_{\ell,m}$ . By the variation of constants formula,  $v$  is given by

$$v(r) = \int_0^\infty G(r, s; \lambda, \ell) s^{\frac{d-1}{2}} e^{-s^2/2} f_{\ell,m}(s) ds$$

where  $G$  is the Green function, i.e.,

$$G(r, s; \lambda, \ell) = -\frac{1}{W(v_0, v_-)(\lambda, \ell)} \begin{cases} v_0(r)v_-(s) & \text{if } r \leq s \\ v_0(s)v_-(r) & \text{if } r \geq s \end{cases}.$$

Here,  $\{v_0, v_-\}$  is a fundamental system for the *homogeneous equation*, that is, Eq. (3.2) with  $f_{\ell,m} = 0$ . The functions  $v_0$  and  $v_-$  need to be chosen in such a way that  $v_0$  is the “good” solution near 0 and  $v_-$  is the “good” solution near  $\infty$ .

The difficulty in obtaining the necessary estimates on  $G$  is the fact that  $v_0$  and  $v_-$  depend on  $\ell$  and  $\lambda$  which are two potentially large parameters. Thus, we need uniform control of  $v_0$  and  $v_-$  for all  $\ell \in \mathbb{N}$

and  $\lambda = d + b + i\omega$  with  $\omega$  large (the parameter  $b$  is fixed). This is a challenging two-parameter asymptotic problem.

Since the potential  $V$  is not known, one cannot hope for explicit expressions for the functions  $v_0$  and  $v_-$ . Consequently, throughout the paper we treat the potential perturbatively, that is, we rewrite the homogeneous version of Eq. (3.2) as

$$v''(r) - r^2 v(r) - \frac{\nu^2 - \frac{1}{4}}{r^2} v(r) - \mu v(r) = -V(r)v(r) \quad (3.3)$$

and for brevity we introduce the parameters  $\mu = \lambda - d = b + i\omega$  and  $\nu = \frac{d}{2} + \ell - 1$ . Our hope is that the right-hand side of Eq. (3.3) is in some sense negligible if  $|\mu|$  is large (this is the only case we are interested in). Consequently, the first step is to solve Eq. (3.3) with  $V = 0$ . Although this equation can be solved explicitly in terms of parabolic cylinder functions, it turns out that the corresponding expressions are still too complicated to proceed. Thus, we do not rely on any kind of asymptotic theory for parabolic cylinder functions but choose a different approach which we now explain.

**3.2. The Liouville-Green transform.** It is expected that the only relevant property of solutions to Eq. (3.3) (with  $V = 0$ ) is their asymptotic behavior as  $r \rightarrow 0+$  and  $r \rightarrow \infty$ , which cannot be terribly complicated. Consequently, it should be possible to add a correction potential  $\tilde{Q}$  (depending on  $\omega$  and  $\nu$ , of course) to both sides of Eq. (3.3) such that

$$v''(r) - r^2 v(r) - \frac{\nu^2 - \frac{1}{4}}{r^2} v(r) - \mu v(r) + \tilde{Q}(r)v(r) = 0$$

has a “simple” fundamental system and the “new” right-hand side

$$-V(r)v(r) + \tilde{Q}(r)v(r)$$

can still be treated perturbatively. The technical device that is used to achieve this is the Liouville-Green transform. We briefly recall how it works which is most easily done by considering a toy problem.

Suppose we are given an equation of the form

$$f''(x) + q(x)f(x) + af(x) = 0 \quad (3.4)$$

where  $a > 0$  is a potentially large parameter. The transformation  $g(\varphi(x)) = \varphi'(x)^{\frac{1}{2}} f(x)$  for an orientation-preserving diffeomorphism  $\varphi$  yields

$$g''(\varphi(x)) + \frac{q(x) + a}{\varphi'(x)^2} g(\varphi(x)) - \frac{Q(x)}{\varphi'(x)^2} g(\varphi(x)) = 0$$

where

$$Q(x) = \frac{1}{2} \frac{\varphi'''(x)}{\varphi'(x)} - \frac{3}{4} \frac{\varphi''(x)^2}{\varphi'(x)^2}$$

is called the Liouville-Green potential. So far, this is a general observation. The transformation becomes useful only if one chooses  $\varphi$  in a clever way, depending on what kind of information one would like to obtain. For instance, if it is possible to choose  $\varphi$  in such a way that  $\frac{q(x)+a}{\varphi'(x)^2} = a$ , one sees that the equation

$$f''(x) + q(x)f(x) + af(x) + Q(x; a)f(x) = 0$$

has the solutions  $\varphi'(x)^{-\frac{1}{2}}e^{\pm ia\varphi(x)}$ . (of course,  $\varphi$  depends on  $a$  but we suppress this). Thus, one rewrites Eq. (3.4) as

$$f''(x) + q(x)f(x) + af(x) + Q(x; a)f(x) = Q(x; a)f(x)$$

and if  $Q$  is small for  $x$  large and  $a$  large, say, one may obtain solutions of Eq. (3.4) of the form  $\varphi'(x)^{-\frac{1}{2}}e^{\pm ia\varphi(x)}[1 + \varepsilon_{\pm}(x, a)]$  where  $\varepsilon_{\pm}(x, a)$  goes to zero as  $x \rightarrow \infty$  or  $a \rightarrow \infty$ . The analysis of solutions of Eq. (3.4) is then reduced to the analysis of the function  $\varphi$  which may be considerably easier.

**3.3. Volterra iterations.** Next, we describe the perturbative treatment of the right-hand side based on Volterra iterations. For simplicity, we stick to the above toy problem Eq. (3.4) and set  $\psi_{\pm}(x; a) := \varphi'(x)^{-\frac{1}{2}}e^{\pm ia\varphi(x)}$ . Suppose we would like to construct a solution to Eq. (3.4) of the form  $\psi_{-}(x; a)[1 + \varepsilon_{-}(x; a)]$  where  $\varepsilon_{-}(x; a)$  vanishes as  $x \rightarrow \infty$ . The variation of constants formula yields the integral equation

$$h(x; a) = 1 + \int_x^{\infty} K(x, y; a)h(y; a)dy \quad (3.5)$$

for the function  $h = 1 + \varepsilon_{-}$ , where

$$K(x, y; a) = \frac{1}{W(a)} \left[ \psi_{-}\psi_{+}(y; a) - \frac{\psi_{+}}{\psi_{-}}(x; a)\psi_{-}(y; a)^2 \right] Q(y; a)$$

and  $W(a) = W(\psi_{-}(\cdot; a), \psi_{+}(\cdot; a)) = -2ia$ . Now suppose

$$m_0 := \int_{x_0}^{\infty} \sup_{x \in (x_0, y)} |K(x, y; a)| dy < \infty$$

for some  $x_0$  and all  $a \geq 1$ . Then the basic theorem on Volterra equations, see e.g. [27], shows that Eq. (3.5) has a solution  $h$  that satisfies  $|h(x; a)| \leq e^{m_0}$  for all  $x \geq x_0$  and  $a \geq 1$ . In a typical situation (for instance if  $Q(x; a)$  decays like  $\langle x \rangle^{-2}$ ) one has an estimate like

$$|K(x, y; a)| \lesssim \langle y \rangle^{-2} a^{-1}$$

for all  $x_0 \leq x \leq y$  and  $a \geq 1$ . This bound implies the existence of  $h$  and the Volterra equation yields the decay

$$|\varepsilon_-(x; a)| = |h(x; a) - 1| \leq e^{m_0} \int_x^\infty |K(x, y; a)| dy \lesssim \langle x \rangle^{-1} a^{-1}.$$

In this way one would obtain a solution to Eq. (3.4) of the form  $\varphi'(x)^{-\frac{1}{2}} e^{-i a \varphi(x)} [1 + O(\langle x \rangle^{-1} a^{-1})]$  and in order to prove bounds that hold uniformly for large  $x$  and large  $a$ , it again suffices to study the function  $\varphi$ .

Another nice feature of Volterra iterations is the fact that the constructed functions inherit differentiability properties of the potential. In a typical situation the potential satisfies symbol-type bounds of the form

$$|\partial_x^k Q(x; a)| \leq C_k \langle x \rangle^{-2-k}, \quad k \in \mathbb{N}_0$$

and these types of bounds are usually inherited by the function  $h$ , cf. Remark 4.4 below.

In the technical part, all our perturbative arguments are based on this scheme and we make free use of the above observations.

**3.4. Construction of fundamental systems.** After this interlude we return to Eq. (3.3). In order to apply the machinery described in Sections 3.2 and 3.3, it is necessary to distinguish a number of cases which we name after the approximating equations. Recall that the relevant parameters are  $\nu = \frac{d}{2} + \ell - 1$  and  $\omega$ , where throughout,  $\mu = \lambda - d = b + i\omega$ . We are only interested in  $\omega$  large (which we always assume) whereas  $\nu$  can be small or large. The parameter  $b$  is fixed and thus not relevant. As a consequence, all implicit constants are allowed to depend on  $b$  (but, of course, *not* on  $\nu$  or  $\omega$ ). Furthermore, we have the variable  $r$  which can be small or large. Depending on the relative location of  $r$ ,  $\omega$ , and  $\nu$ , we move different terms in Eq. (3.3) to the right-hand side, apply a suitable Liouville-Green transform, and perform a perturbative construction as outlined in Sections 3.2 and 3.3.

(1) Small angular momenta:  $\nu \leq \nu_0$  for some sufficiently large absolute constant  $\nu_0 > 0$ .

(a) Weber case:  $r \geq 1$ , Proposition 4.3. We rewrite Eq. (3.3) as

$$\begin{aligned} v''(r) - r^2 v(r) - \frac{\nu^2}{r^2} v(r) - \mu v(r) \\ = -\frac{1}{4r^2} v(r) - V(r) v(r). \end{aligned}$$

The dominant contribution comes from a Weber-type equation. We construct a fundamental system  $\{v_-, v_+\}$  of the

form

$$v_{\pm}(r, \omega) = \frac{1}{\sqrt{2}} \mu^{-\frac{1}{4}} \xi'(\mu^{-\frac{1}{2}} r)^{-\frac{1}{2}} e^{\pm \mu \xi(\mu^{-1/2} r)} [1 + \varepsilon_{\pm}(r, \omega)]$$

where  $\varepsilon_{\pm}$  are complex-valued functions that satisfy the bound

$$|\varepsilon_{\pm}(r, \omega)| \leq C r^{-1} \omega^{-\frac{1}{2}}$$

for all  $r \geq 1$ ,  $\omega \geq \omega_0$  (where  $\omega_0$  is a sufficiently large absolute constant),  $\nu \geq 0$ , and some absolute constant  $C > 0$ . Of course,  $v_{\pm}$  and  $\varepsilon_{\pm}$  also depend on  $\nu$  but in the domain  $\nu \leq \nu_0$  this dependence is inessential and we suppress it in the notation. In the sequel we will use the compact notation  $\varepsilon_{\pm}(r, \omega) = O_{\mathbb{C}}(r^{-1} \omega^{-\frac{1}{2}} \nu^0)$  which carries all the relevant information. The function  $\xi$  is given in closed form as an integral and its analysis is in principle straightforward.

- (b) Hankel case:  $c\omega^{-\frac{1}{2}} \leq r \leq 1$  where  $c > 0$  is a sufficiently large absolute constant, Lemma 4.6. Since  $r$  is bounded, we can move the Weber term to the right-hand side and consider

$$v''(r) - \frac{\nu^2 - \frac{1}{4}}{r^2} v(r) - \mu v(r) = r^2 v(r) - V(r) v(r).$$

Thus, the dominant contribution comes from a Bessel equation and we construct a fundamental system by perturbing Hankel functions.

- (c) Bessel case:  $0 < r < c\omega^{-\frac{1}{2}}$ , Lemma 4.5. This is similar to the Hankel case but we replace the Hankel functions by Bessel functions in order to gain control near  $r = 0$ .
- (2) Large angular momenta:  $\nu \geq \nu_0$ .
- (a) Weber case:  $r \geq 1$ , Proposition 4.3. This case is in fact already handled by the small angular momenta Weber case since it turns out that for the fundamental system  $\{v_{\pm}\}$  one has good control for all  $\nu \geq 0$  and not just  $\nu \leq \nu_0$ .
- (b) Bessel case:  $0 < r \leq 1$ , Lemma 4.12. This is reminiscent of the classical asymptotic theory for Bessel functions. We rewrite Eq. (3.3) as

$$\begin{aligned} v''(r) - \frac{\nu^2}{r^2} v(r) - \mu v(r) \\ = -\frac{1}{4r^2} v(r) + r^2 v(r) - V(r) v(r) \end{aligned}$$

and construct a fundamental system by perturbing the asymptotic form of Bessel functions for large parameters.

The representations of solutions to Eq. (3.3) in the various regimes are then used to compute the Wronskian  $W(v_0, v_-)$  and to estimate the Green function for large  $\omega$ , uniformly in  $\nu$ . The bound on the reduced resolvent can then be obtained in a straightforward manner.

**3.5. A remark on notation.** In the technical part we make extensive use of the “ $O_{\mathbb{C}}$ -notation”. In this respect we would like to reiterate that there is no hidden dependence of any kind on the relevant parameters  $\omega$  and  $\nu$ . On the contrary, it is of course precisely the point of our whole construction to track the dependence on  $\omega$  and  $\nu$  explicitly through all computations. Due to the complexity of the calculations, this is only possible with an economic notation that keeps track of the relevant information but suppresses all the irrelevant details. In particular, as is standard in many branches of analysis, we hardly ever denote absolute constants explicitly. The dependence on  $\omega$  and  $\nu$ , on the other hand, is always explicitly stated, even if it is not relevant in the particular context. For instance, we use the notation  $O_{\mathbb{C}}(\omega^{-\frac{1}{2}}\nu^0)$  for a complex-valued function that depends on  $\omega$ ,  $\nu$  and which is bounded by  $C\omega^{-\frac{1}{2}}$  for some absolute constant  $C > 0$  in the relevant range of  $\omega$  and  $\nu$  (which is also stated explicitly).

#### 4. CONSTRUCTION OF FUNDAMENTAL SYSTEMS

**4.1. Reduction to normal form.** In order to construct the reduced resolvent  $R_{\ell,m}(\lambda)$  we have to solve the equation

$$\begin{aligned} u''(r) + \frac{d-1}{r}u'(r) - 2ru'(r) - \frac{\ell(\ell+d-2)}{r^2}u(r) \\ + V(r)u(r) - \lambda u(r) = -f(r). \end{aligned} \quad (4.1)$$

Setting  $u(r) := r^{-\frac{d-1}{2}}e^{r^2/2}v(r)$  yields the normal form equation

$$\begin{aligned} v''(r) - r^2v(r) - \frac{(d+2\ell-1)(d+2\ell-3)}{4r^2}v(r) - (\lambda - d)v(r) \\ + V(r)v(r) = -r^{\frac{d-1}{2}}e^{-r^2/2}f(r). \end{aligned} \quad (4.2)$$

Consequently, our first task is to construct a fundamental system for

$$\begin{aligned} v''(r) - r^2v(r) - \frac{(d+2\ell-1)(d+2\ell-3)}{4r^2}v(r) - (\lambda - d)v(r) \\ + V(r)v(r) = 0. \end{aligned} \quad (4.3)$$

We set  $\nu = \frac{d}{2} + \ell - 1$ ,  $\mu = \lambda - d$  and rewrite Eq. (4.3) as

$$v''(r) - r^2v(r) - \frac{\nu^2}{r^2}v(r) - \mu v(r) = -\frac{1}{4r^2}v(r) - V(r)v(r). \quad (4.4)$$

**4.2. A fundamental system away from the center.** As suggested by the notation, we intend to treat the right-hand side of Eq. (4.4) perturbatively. Thus, for the moment we set it to zero and note that the rescaling  $w(y) = v(\mu^{\frac{1}{2}}y)$  with  $\mu > 0$  yields the equation

$$w''(y) - \mu^2(1 + y^2)w(y) - \frac{\nu^2}{y^2}w(y) = 0.$$

The Liouville-Green transform  $\tilde{w}(\xi(y)) = |\xi'(y)|^{\frac{1}{2}}w(y)$  leads to

$$\tilde{w}''(\xi(y)) - \frac{\mu^2(1 + y^2) + \frac{\nu^2}{y^2}}{\xi'(y)^2}\tilde{w}(\xi(y)) - \frac{Q(y)}{\xi'(y)^2}\tilde{w}(\xi(y)) = 0$$

with

$$Q(y) = \frac{1}{2} \frac{\xi'''(y)}{\xi'(y)} - \frac{3}{4} \frac{\xi''(y)^2}{\xi'(y)^2}.$$

Consequently, it is reasonable to look for a diffeomorphism  $\xi$  that satisfies

$$\frac{\mu^2(1 + y^2) + \frac{\nu^2}{y^2}}{\xi'(y)^2} = \mu^2$$

or

$$\xi'(y) = \sqrt{1 + y^2 + \frac{\nu^2}{\mu^2 y^2}}$$

and thus,

$$\xi(y) := \int_{\mu^{-1/2}}^y \sqrt{1 + s^2 + \frac{\nu^2}{\mu^2 s^2}} ds$$

is a possible choice (the lower bound is arbitrary but this choice turns out to be convenient). A more precise notation would be  $\xi(y; \mu, \nu)$  but we refrain from using this in order to keep the equations shorter. This sloppiness comes at the price of strange-looking identities like  $\xi(\mu^{-\frac{1}{2}}) = 0$ . For the Liouville-Green potential we infer

$$Q(y) = \frac{2y^2 - 3y^4 + 6\frac{\alpha^2}{y^2} + 18y^2\frac{\alpha^2}{y^2} + \frac{\alpha^4}{y^4}}{4y^2(1 + y^2 + \frac{\alpha^2}{y^2})^2}, \quad \alpha := \frac{\nu}{\mu}. \quad (4.5)$$

By construction, the equation

$$w''(y) - \mu^2(1 + y^2)w(y) - \frac{\nu^2}{y^2}w(y) + Q(y)w(y) = 0 \quad (4.6)$$

transforms into

$$\tilde{w}''(\xi(y)) - \mu^2\tilde{w}(\xi(y)) = 0$$

for  $\tilde{w}(\xi(y)) = \xi'(y)^{\frac{1}{2}}w(y)$  and thus, Eq. (4.6) has the fundamental system

$$\xi'(y)^{-\frac{1}{2}}e^{\pm\mu\xi(y)}.$$



By setting  $y = \mu^{-\frac{1}{2}}r$  and  $w(y) = v(\mu^{\frac{1}{2}}y)$ , Eq. (4.6) transforms into

$$v''(r) - r^2v(r) - \frac{\nu^2}{r^2}v(r) + \mu^{-1}Q(\mu^{-\frac{1}{2}}r)v(r) - \mu v(r) = 0 \quad (4.7)$$

with the fundamental system

$$\xi'(\mu^{-\frac{1}{2}}r)^{-\frac{1}{2}}e^{\pm\mu\xi(\mu^{-1/2}r)}. \quad (4.8)$$

This suggests to rewrite Eq. (4.4) as

$$\begin{aligned} v''(r) - r^2v(r) - \frac{\nu^2}{r^2}v(r) + \mu^{-1}Q(\mu^{-\frac{1}{2}}r)v(r) - \mu v(r) \\ = \mu^{-1}Q(\mu^{-\frac{1}{2}}r)v(r) - \frac{1}{4r^2}v(r) - V(r)v(r) \end{aligned} \quad (4.9)$$

and the hope is to treat the right-hand side perturbatively.

**4.2.1. Analysis of  $\xi$  and  $Q$ .** So far we were dealing with  $\mu > 0$  but we are actually interested in  $\mu = b + i\omega$  for fixed  $b \in \mathbb{R}$  and  $\omega$  large. For  $\mu > 0$  the function  $\xi$  can be written as

$$\begin{aligned} \xi(\mu^{-\frac{1}{2}}r) &= \int_{\mu^{-1/2}}^{\mu^{-1/2}r} \sqrt{1 + s^2 + \frac{\nu^2}{\mu^2 s^2}} ds \\ &= \mu^{-\frac{1}{2}} \int_1^r \sqrt{1 + \frac{s^2}{\mu} + \frac{\nu^2}{\mu s^2}} ds \end{aligned} \quad (4.10)$$

and the last expression makes perfect sense even for  $\mu = b + i\omega$  as a contour integral of a holomorphic function (the argument of the square root stays in  $\mathbb{C} \setminus (-\infty, 0]$  for all  $s \geq 1$ ). We note that  $\sqrt{\cdot}$  always means the principal branch of the complex square root, holomorphic in  $\mathbb{C} \setminus (-\infty, 0]$  and explicitly given by

$$\sqrt{z} = \frac{1}{\sqrt{2}}\sqrt{|z| + \operatorname{Re} z} + \frac{i \operatorname{sgn}(\operatorname{Im} z)}{\sqrt{2}}\sqrt{|z| - \operatorname{Re} z}. \quad (4.11)$$

As a direct consequence of the explicit formula (4.11) we have  $\sqrt{z^2} = z$  and  $|\sqrt{z}| = \sqrt{|z|}$  for all  $z \in \mathbb{C} \setminus (-\infty, 0]$ . Furthermore, the formula

$$\sqrt{z}\sqrt{w} = \sqrt{zw}$$

is valid at least if  $\operatorname{Re} z \geq 0$  and  $\operatorname{Re} w > 0$ .

Based on the explicit expression (4.5), the Liouville-Green potential  $\mu^{-1}Q(\mu^{-\frac{1}{2}}r)$  has a straightforward analytic continuation to  $\mu = b + i\omega$ . As a consequence, (4.8) is a fundamental system for Eq. (4.7) also in the complex case  $\mu = b + i\omega$ . We remark that in general, all functions depend on the parameter  $\nu$  and this dependence is crucial. However, for the sake of readability we usually suppress it in the notation.

The following bound shows that the right-hand side of Eq. (4.9) can indeed be treated perturbatively.

**Lemma 4.1.** *Let  $\mu = b + i\omega$  where  $b \in \mathbb{R}$  is fixed. Then we have the bound*

$$|\mu^{-1}Q(\mu^{-\frac{1}{2}}r)| \lesssim r^{-2}$$

for all  $r > 0$ ,  $\omega \gg 1$ , and  $\nu \geq 0$ .

*Proof.* We set  $\alpha = \frac{\nu}{\mu}$ . For all  $\alpha, y \in \mathbb{C}$  we have the bound

$$|Q(y)| \lesssim \frac{|\frac{\alpha}{y}|^2 + |\frac{\alpha}{y}|^4}{|y|^2|1 + y^2 + \frac{\alpha^2}{y^2}|^2} + \frac{|y|^2 + |y|^4}{|y|^2|1 + y^2 + \frac{\alpha^2}{y^2}|^2}. \quad (4.12)$$

In order to estimate the denominator, we use the bound

$$|1 + \frac{x}{\mu}|^2 \geq \frac{1}{2}(1 + \frac{x^2}{|\mu|^2}), \quad x \in \mathbb{R} \quad (4.13)$$

which holds for  $\omega \gg 1$  as a consequence of

$$\begin{aligned} |1 + \frac{x}{\mu}|^2 &= 1 + 2x\operatorname{Re}(\mu^{-1}) + \frac{x^2}{|\mu|^2} \\ &= 1 + \frac{2b}{\sqrt{b^2 + \omega^2}} \frac{x}{\sqrt{b^2 + \omega^2}} + \frac{x^2}{b^2 + \omega^2} \\ &\geq 1 - \frac{2b^2}{b^2 + \omega^2} + \frac{1}{2} \frac{x^2}{b^2 + \omega^2}. \end{aligned}$$

Thus, Eqs. (4.13) and (4.12) imply

$$|Q(\mu^{-\frac{1}{2}}r)| \lesssim |\mu|r^{-2}$$

which yields the claim.  $\square$

The following representation of  $\xi$  is crucial and contains in fact all the information on  $\xi$  we are going to use.

**Lemma 4.2.** *Let  $\mu = b + i\omega$  where  $b \in \mathbb{R}$  is fixed. Then we have the representation*

$$\operatorname{Re}[\mu\xi(\mu^{-\frac{1}{2}}r)] = \frac{1}{2}r^2 + \frac{b}{2}\log\langle\mu^{-\frac{1}{2}}r\rangle + \varphi(r; \omega, \nu)$$

where  $\partial_r\varphi(r; \omega, \nu) \geq 0$  for all  $r > 0$ ,  $\omega \gg 1$ , and  $\nu \geq 0$ .

*Proof.* We only prove the case  $b \geq 0$ . From Eq. (4.10) we obtain

$$\begin{aligned} \partial_r \operatorname{Re}[\mu\xi(\mu^{-\frac{1}{2}}r)] &= \operatorname{Re}[\mu\partial_r\xi(\mu^{-\frac{1}{2}}r)] = \operatorname{Re}\sqrt{\mu + r^2 + \frac{\nu^2}{r^2}} \\ &= \frac{1}{\sqrt{2}}\sqrt{|\mu + r^2 + \frac{\nu^2}{r^2}| + b + r^2 + \frac{\nu^2}{r^2}} \\ &\geq \sqrt{b + r^2} \end{aligned}$$

since

$$\begin{aligned} |\mu + r^2 + \frac{\nu^2}{r^2}|^2 &= |\mu|^2 + 2b(r^2 + \frac{\nu^2}{r^2}) + (r^2 + \frac{\nu^2}{r^2})^2 \\ &\geq b^2 + 2br^2 + r^4 = (b + r^2)^2. \end{aligned}$$

Consequently, we find

$$\begin{aligned}
\partial_r \varphi(r; \omega, \nu) &= \partial_r \left\{ \operatorname{Re} [\mu \xi(\mu^{-\frac{1}{2}} r)] - \frac{1}{2} r^2 - \frac{b}{2} \log \langle \mu^{-\frac{1}{2}} r \rangle \right\} \\
&\geq \sqrt{b + r^2} - r - \frac{\frac{b}{2} |\mu|^{-\frac{1}{2}} |\mu|^{-\frac{1}{2}} r}{\langle \mu^{-\frac{1}{2}} r \rangle^2} \\
&\geq \frac{b}{\sqrt{b + r^2} + r} - \frac{\frac{b}{2} |\mu|^{-\frac{1}{2}}}{\langle \mu^{-\frac{1}{2}} r \rangle} \\
&= \frac{b |\mu|^{-\frac{1}{2}}}{\sqrt{\frac{b}{|\mu|} + \frac{r^2}{|\mu|} + |\mu|^{-\frac{1}{2}} r}} - \frac{\frac{b}{2} |\mu|^{-\frac{1}{2}}}{\langle \mu^{-\frac{1}{2}} r \rangle} \\
&\geq \frac{\frac{b}{2} |\mu|^{-\frac{1}{2}}}{\langle \mu^{-\frac{1}{2}} r \rangle} - \frac{\frac{b}{2} |\mu|^{-\frac{1}{2}}}{\langle \mu^{-\frac{1}{2}} r \rangle} = 0.
\end{aligned}$$

□

4.2.2. *A fundamental system for Eq. (4.9).* The functions

$$v_0^\pm(r, \omega) := \frac{1}{\sqrt{2}} \mu^{-\frac{1}{4}} \xi'(\mu^{-\frac{1}{2}} r)^{-\frac{1}{2}} e^{\pm \mu \xi(\mu^{-1/2} r)}, \quad \mu = b + i\omega$$

are solutions to Eq. (4.7). In order to compute their Wronskian, we note that for any holomorphic function  $f$  and  $r \in \mathbb{R}$  we have the chain rule  $\partial_r f(rz) = z f'(rz)$ ,  $z \in \mathbb{C}$ . Consequently, we obtain

$$\partial_r v_0^\pm(r, \omega) = \frac{1}{\sqrt{2}} \mu^{-\frac{1}{4}} \left[ \partial_r [\xi'(\mu^{-\frac{1}{2}} r)^{-\frac{1}{2}}] \pm \mu^{\frac{1}{2}} \xi'(\mu^{-\frac{1}{2}} r)^{\frac{1}{2}} \right] e^{\pm \mu \xi(\mu^{-1/2} r)}$$

which yields

$$W(v_0^-(\cdot, \omega), v_0^+(\cdot, \omega)) = 1. \quad (4.14)$$

With this information at hand we are ready to construct a fundamental system for Eq. (4.9). Note that by Lemma 4.1, the right-hand side of Eq. (4.9) is  $O_{\mathbb{C}}(r^{-2} \omega^0 \nu^0) v(r)$ .

**Proposition 4.3.** *Let  $\mu = b + i\omega$  where  $b \in \mathbb{R}$  is fixed. Then Eq. (4.9) has a fundamental system  $\{v_-, v_+\}$  of the form*

$$\begin{aligned}
v_\pm(r, \omega) &= v_0^\pm(r, \omega) [1 + O_{\mathbb{C}}(r^{-1} \omega^{-\frac{1}{2}} \nu^0)] \\
&= \frac{1}{\sqrt{2}} \mu^{-\frac{1}{4}} \xi'(\mu^{-\frac{1}{2}} r)^{-\frac{1}{2}} e^{\pm \mu \xi(\mu^{-1/2} r)} [1 + O_{\mathbb{C}}(r^{-1} \omega^{-\frac{1}{2}} \nu^0)]
\end{aligned}$$

for all  $r \geq 1$ ,  $\omega \gg 1$ , and  $\nu \geq 0$ .

*Proof.* We only treat the case  $b \geq 0$ . In order to construct a solution  $v_-(r, \omega)$  of Eq. (4.9) that behaves like  $v_0^-(r, \omega)$  as  $r \rightarrow \infty$ , we consider

the integral equation

$$\begin{aligned} v_-(r, \omega) = & v_0^-(r, \omega) + v_0^+(r, \omega) \int_r^\infty v_0^-(s, \omega) O_{\mathbb{C}}(s^{-2}\omega^0\nu^0) v_-(s, \omega) ds \\ & - v_0^-(r, \omega) \int_r^\infty v_0^+(s, \omega) O_{\mathbb{C}}(s^{-2}\omega^0\nu^0) v_-(s, \omega) ds. \end{aligned}$$

We note that  $|v_0^\pm(r, \omega)| > 0$  for all  $r > 0$  and set  $h_- := \frac{v_-}{v_0^-}$  which yields the Volterra equation

$$h_-(r, \omega) = 1 + \int_r^\infty K(r, s, \omega) h_-(s, \omega) ds$$

where

$$K(r, s, \omega) := \left[ \frac{v_0^+(r, \omega)}{v_0^-(r, \omega)} v_0^-(s, \omega)^2 - v_0^- v_0^+(s, \omega) \right] O_{\mathbb{C}}(s^{-2}\omega^0\nu^0).$$

Now we prove pointwise bounds on  $K$ . We have

$$\begin{aligned} |v_0^- v_0^+(s, \omega)| &= \frac{1}{2} |\mu|^{-\frac{1}{2}} |\xi'(\mu^{-\frac{1}{2}} s)|^{-1} = \frac{1}{2} |\mu|^{-\frac{1}{2}} \left| 1 + \frac{1}{\mu} (s^2 + \frac{\nu^2}{s^2}) \right|^{-\frac{1}{2}} \\ &\lesssim \omega^{-\frac{1}{2}}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \left| \frac{v_0^+(r, \omega)}{v_0^-(r, \omega)} v_0^-(s, \omega)^2 \right| &= \frac{1}{2} |\mu|^{-\frac{1}{2}} \left| e^{2\mu\xi(\mu^{-1/2}r)} \xi'(\mu^{-\frac{1}{2}}s)^{-1} e^{-2\mu\xi(\mu^{-1/2}s)} \right| \\ &\lesssim \omega^{-\frac{1}{2}} e^{-2\operatorname{Re}[\mu\xi(\mu^{-1/2}s) - \mu\xi(\mu^{-1/2}r)]} \\ &\lesssim \omega^{-\frac{1}{2}} \langle \mu^{-\frac{1}{2}}r \rangle^b \langle \mu^{-\frac{1}{2}}s \rangle^{-b} e^{-(s^2-r^2)} \\ &\quad \times e^{-2[\varphi(s; \omega, \nu) - \varphi(r; \omega, \nu)]} \\ &\lesssim \omega^{-\frac{1}{2}} \end{aligned}$$

for all  $0 \leq r \leq s$ ,  $\omega \gg 1$ , and  $\nu \geq 0$  by Lemma 4.2. As a consequence, we infer the estimate  $|K(r, s, \omega)| \lesssim \omega^{-\frac{1}{2}} s^{-2}$  which implies

$$\int_1^\infty \sup_{r \in [1, s]} |K(r, s, \omega)| ds \lesssim \omega^{-\frac{1}{2}} \int_1^\infty s^{-2} ds \lesssim 1$$

and the standard result on Volterra equations (see Remark 4.4 below) yields the existence of a solution  $h_-$  with the bound  $|h_-(r, \omega) - 1| \lesssim r^{-1} \omega^{-\frac{1}{2}}$ .

In order to construct the solution  $v_+$  we note that  $|v_-(r, \omega)| > 0$  for all  $r \geq 0$  and  $\omega \gg 1$ . Consequently, a second solution of Eq. (4.9) is given by

$$v_+(r, \omega) := v_-(r, \omega) \left[ \frac{v_0^+(1, \omega)}{v_0^-(1, \omega)} + \int_1^r v_-(s, \omega)^{-2} ds \right].$$

We set  $h_+ := \frac{v_+}{v_0^+}$ . The identity

$$v_0^-(r, \omega) \int_1^r v_0^-(s, \omega)^{-2} ds = v_0^+(r, \omega) - \frac{v_0^+(1, \omega)}{v_0^-(1, \omega)} v_0^-(r, \omega),$$

which follows from  $W(v_0^-(\cdot, \omega), v_0^+(\cdot, \omega)) = 1$ , then yields

$$\begin{aligned} h_+(r, \omega) &= 1 + O_{\mathbb{C}}(r^{-1} \omega^{-\frac{1}{2}} \nu^0) \\ &\quad - \frac{v_-(r, \omega)}{v_0^+(r, \omega)} \int_1^r v_0^-(s, \omega)^{-2} O_{\mathbb{C}}(s^{-1} \omega^{-\frac{1}{2}} \nu^0) ds. \end{aligned}$$

Now recall that

$$\begin{aligned} v_0^-(s, \omega)^{-2} &= 2\mu^{\frac{1}{2}} \xi'(\mu^{-\frac{1}{2}} s) e^{2\mu\xi(\mu^{-1/2} s)} \\ &= \partial_s e^{2\mu\xi(\mu^{-1/2} s)} \end{aligned}$$

and thus, an integration by parts yields

$$\begin{aligned} h_+(r, \omega) &= 1 + O_{\mathbb{C}}(r^{-1} \omega^{-\frac{1}{2}} \nu^0) \\ &\quad + e^{-2\mu\xi(\mu^{-1/2} r)} [1 + O_{\mathbb{C}}(r^{-1} \omega^{-\frac{1}{2}} \nu^0)] \\ &\quad \times \left[ e^{2\mu\xi(\mu^{-1/2} r)} O_{\mathbb{C}}(r^{-1} \omega^{-\frac{1}{2}} \nu^0) - e^{2\mu\xi(\mu^{-1/2})} O_{\mathbb{C}}(\omega^{-\frac{1}{2}} \nu^0) \right. \\ &\quad \left. + \int_1^r e^{2\mu\xi(\mu^{-1/2} s)} O_{\mathbb{C}}(s^{-2} \omega^{-\frac{1}{2}} \nu^0) ds \right] \\ &= 1 + O_{\mathbb{C}}(r^{-1} \omega^{-\frac{1}{2}} \nu^0) \end{aligned}$$

since

$$\left| e^{-2[\mu\xi(\mu^{-1/2} r) - \mu\xi(\mu^{-1/2} s)]} \right| \lesssim e^{-(r^2 - s^2)}$$

for all  $r \geq s$ ,  $\omega \gg 1$ , and  $\nu \geq 0$ , see Lemma 4.2.  $\square$

*Remark 4.4.* The Volterra equation

$$f(x) = g(x) + \int_x^\infty K(x, y) f(y) dy$$

has a solution  $f \in L^\infty(a, \infty)$  if  $g \in L^\infty(a, \infty)$  and

$$\int_a^\infty \sup_{x \in (a, y)} |K(x, y)| dy < \infty,$$

see e.g. [27]. In addition,  $f$  inherits differentiability properties from  $g$  and  $K$ . For instance, if the kernel  $K$  is of the form  $K(x, y) = e^{\phi(x) - \phi(y)} \tilde{K}(x, y)$ , where  $\phi : (0, \infty) \rightarrow (0, \infty)$  is an orientation-preserving

diffeomorphism, and the functions  $\phi$ ,  $\tilde{K}$ , and  $g$  behave like symbols<sup>1</sup>, then  $f$  has the same property. This follows by a simple induction from the identity

$$\begin{aligned} f(x) &= g(x) + \int_x^\infty e^{\phi(x)-\phi(y)} \tilde{K}(x, y) f(y) dy \\ &= g(x) + \int_0^\infty e^{-y'} \tilde{K}(x, \phi^{-1}(y' + \phi(x))) \\ &\quad \times f(\phi^{-1}(y' + \phi(x))) \frac{dy'}{\phi'(\phi^{-1}(y' + \phi(x)))} \end{aligned}$$

which shows that  $x$ -derivatives hit only terms that behave like symbols.

**4.3. A fundamental system near the center.** Next, we consider Eq. (4.3) for  $0 < r \leq 1$ . In this case we move the term  $r^2 v(r)$  to the right-hand side and treat it perturbatively, i.e., we rewrite Eq. (4.4) as

$$v''(r) - \frac{\nu^2 - \frac{1}{4}}{r^2} v(r) - \mu v(r) = O_{\mathbb{C}}(\langle r \rangle^2) v(r). \quad (4.15)$$

We consider the “homogeneous” version

$$v''(r) - \frac{\nu^2 - \frac{1}{4}}{r^2} v(r) - \mu v(r) = 0 \quad (4.16)$$

and rescale by introducing  $v(r) = w(\mu^{\frac{1}{2}} r)$  with  $\mu > 0$ . This rescaling yields the modified Bessel equation

$$w''(y) - \frac{\nu^2 - \frac{1}{4}}{y^2} w(y) - w(y) = 0$$

where  $y = \mu^{\frac{1}{2}} r$ . A fundamental system for this equation is given by

$$\sqrt{y} J_\nu(i y), \quad \sqrt{y} Y_\nu(i y)$$

where  $J_\nu$  and  $Y_\nu$  are the standard Bessel functions, see e.g. [22, 23]. Consequently, Eq. (4.16) has the fundamental system

$$\sqrt{r} J_\nu(i \mu^{\frac{1}{2}} r), \quad \sqrt{r} Y_\nu(i \mu^{\frac{1}{2}} r).$$

**Lemma 4.5.** *Let  $\nu_0 > 0$  and  $\mu = b + i\omega$  where  $b \in \mathbb{R}$  is fixed. Furthermore, fix  $c \geq 1$ . Then Eq. (4.15) has a fundamental system  $\{v_0, v_1\}$  of*

---

<sup>1</sup> A function  $f \in C^k(I)$ ,  $I \subset \mathbb{R}$  an interval, is said to behave like a symbol if  $|f^{(j)}(x)| \leq C|x|^{\gamma-j}$  for some  $\gamma \in \mathbb{R}$ , all  $0 \leq j \leq k$ , and all  $x \in I$ . A similar notion applies to functions of several variables. As a consequence of the chain rule, symbol behavior is preserved under the usual algebraic operations.

the form

$$\begin{aligned} v_0(r, \omega) &= \sqrt{r} J_\nu(\mathbf{i} \mu^{\frac{1}{2}} r) [1 + O_{\mathbb{C}}(r^0 \omega^{-\frac{1}{2}})] \\ v_1(r, \omega) &= \sqrt{r} [Y_\nu(\mathbf{i} \mu^{\frac{1}{2}} r) + O_{\mathbb{C}}(\omega^0) J_\nu(\mathbf{i} \mu^{\frac{1}{2}} r)] [1 + O_{\mathbb{C}}(r^0 \omega^{-\frac{1}{2}})] \end{aligned}$$

for all  $r \in (0, c\omega^{-\frac{1}{2}}]$ ,  $\omega \gg c^2$ , and  $\nu \in [0, \nu_0]$ .

*Proof.* The Bessel function  $J_\nu$  is holomorphic in  $\mathbb{C} \setminus (-\infty, 0]$  and hence, by analytic continuation,

$$\psi_0(r, \omega) := \sqrt{r} J_\nu(\mathbf{i} \mu^{\frac{1}{2}} r)$$

is a solution to Eq. (4.16) with  $\mu = b + \mathbf{i} \omega$ . Recall that all zeros of  $J_\nu$  are real ([22], p. 245, Theorem 6.2) and therefore,  $|\psi_0(r, \omega)| > 0$  for all  $r > 0$ . Thus, we may set

$$\psi_1(r, \omega) := -\psi_0(r, \omega) \int_r^{c\omega^{-1/2}} \psi_0(s, \omega)^{-2} ds$$

and this yields another solution to Eq. (4.16). For the Wronskian of  $\psi_0$  and  $\psi_1$  we have  $W(\psi_0(\cdot, \omega), \psi_1(\cdot, \omega)) = 1$  which shows that  $\{\psi_0, \psi_1\}$  is a fundamental system for Eq. (4.16). For notational convenience we set  $\tilde{\psi}_1(r, \omega) := \sqrt{r} Y_\nu(\mathbf{i} \mu^{\frac{1}{2}} r)$ . From  $W(J_\nu, Y_\nu)(z) = \frac{2}{\pi z}$  we infer  $W(\psi_0(\cdot, \omega), \tilde{\psi}_1(\cdot, \omega)) = \frac{2}{\pi}$  and thus,  $\{\psi_0, \tilde{\psi}_1\}$  is another fundamental system for Eq. (4.16). Consequently, we have the connection formula

$$\psi_1(r, \omega) = \frac{W(\psi_1(\cdot, \omega), \tilde{\psi}_1(\cdot, \omega))}{W(\psi_0(\cdot, \omega), \tilde{\psi}_1(\cdot, \omega))} \psi_0(r, \omega) + \frac{W(\psi_1(\cdot, \omega), \psi_0(\cdot, \omega))}{W(\tilde{\psi}_1(\cdot, \omega), \psi_0(\cdot, \omega))} \tilde{\psi}_1(r, \omega)$$

and by evaluation at  $r = c\omega^{-\frac{1}{2}}$  we find

$$W(\psi_1(\cdot, \omega), \tilde{\psi}_1(\cdot, \omega)) = -\frac{\tilde{\psi}_1(c\omega^{-\frac{1}{2}}, \omega)}{\psi_0(c\omega^{-\frac{1}{2}}, \omega)} = -\frac{Y_\nu(\mathbf{i} c\mu^{\frac{1}{2}} \omega^{-\frac{1}{2}})}{J_\nu(\mathbf{i} c\mu^{\frac{1}{2}} \omega^{-\frac{1}{2}})} = O_{\mathbb{C}}(\omega^0).$$

This yields the representation

$$\psi_1(r, \omega) = \frac{\pi}{2} \sqrt{r} Y_\nu(\mathbf{i} \mu^{\frac{1}{2}} r) + O_{\mathbb{C}}(\omega^0) \sqrt{r} J_\nu(\mathbf{i} \mu^{\frac{1}{2}} r).$$

In order to construct  $v_0$ , we have to look for a solution of the integral equation

$$\begin{aligned} v_0(r, \omega) &= \psi_0(r, \omega) - \psi_0(r, \omega) \int_0^r \psi_1(s, \omega) O_{\mathbb{C}}(\langle s \rangle^2) v_0(s, \omega) ds \\ &\quad + \psi_1(r, \omega) \int_0^r \psi_0(s, \omega) O_{\mathbb{C}}(\langle s \rangle^2) v_0(s, \omega) ds. \end{aligned}$$

Consequently, upon setting  $v_0 = \psi_0 h_0$ , we obtain the Volterra equation

$$h_0(r, \omega) = 1 + \int_0^r K(r, s, \omega) h_0(s, \omega) ds \quad (4.17)$$

with the kernel

$$K(r, s, \omega) = \left[ \frac{\psi_1(r, \omega)}{\psi_0(r, \omega)} \psi_0(s, \omega)^2 - \psi_0(s, \omega) \psi_1(s, \omega) \right] O_{\mathbb{C}}(\langle s \rangle^2).$$

We have  $|\psi_0(r, \omega)| \simeq \omega^{\frac{\nu}{2}} r^{\frac{1}{2} + \nu}$  and<sup>2</sup>  $|\psi_1(r, \omega)| \simeq \omega^{-\frac{\nu}{2}} r^{\frac{1}{2} - \nu}$  for  $0 < r \leq c\omega^{-\frac{1}{2}}$  and  $\omega \gg c^2$  and thus,  $|K(r, s, \omega)| \lesssim s$  for  $0 \leq s \leq r \leq c\omega^{-\frac{1}{2}}$ . This implies

$$\int_0^{c\omega^{-1/2}} \sup_{r \in [s, c\omega^{-1/2}]} |K(r, s, \omega)| ds \lesssim \omega^{-\frac{1}{2}}$$

and the standard result on Volterra equations yields a solution of Eq. (4.17) with the bound  $|h_0(r, \omega) - 1| \lesssim \omega^{-\frac{1}{2}}$ . Since  $|v_0(r, \omega)| > 0$  for all  $r \in (0, c\omega^{-\frac{1}{2}}]$  provided  $\omega$  is sufficiently large, a second solution  $v_1$  is given by

$$v_1(r, \omega) = -\frac{2}{\pi} v_0(r, \omega) \int_r^{c\omega^{-1/2}} v_0(s, \omega)^{-2} ds$$

and it is straightforward to verify that it is indeed of the stated form.  $\square$

Unfortunately, we cannot directly glue together the fundamental systems from Proposition 4.3 and Lemma 4.5. As a consequence, we still require another fundamental system which allows us to bridge the gap  $r \in [c\omega^{-\frac{1}{2}}, 1]$ . The latter is obtained by perturbing Hankel functions  $H_{\nu}^{\pm} = J_{\nu} \pm i Y_{\nu}$ .

**Lemma 4.6.** *Let  $\nu_0 > 0$  and  $\mu = b + i\omega$  where  $b \in \mathbb{R}$  is fixed. Then there exists a fundamental system  $\{\tilde{v}_-, \tilde{v}_+\}$  for Eq. (4.15) of the form*

$$\tilde{v}_{\pm}(r, \omega) = \sqrt{r} H_{\nu}^{\mp}(i\mu^{\frac{1}{2}} r) [1 + O_{\mathbb{C}}(r^0 \omega^{-\frac{1}{2}})]$$

for all  $r \in [c\omega^{-\frac{1}{2}}, 1]$ ,  $\omega \gg c^2$ , and  $\nu \in [0, \nu_0]$ , provided  $c \geq 1$  is sufficiently large.

*Proof.* We set

$$\psi_{\pm}(r, \omega) := c_{\nu}^{\pm} \sqrt{r} H_{\nu}^{\mp}(i\mu^{\frac{1}{2}} r), \quad \mu = b + i\omega$$

where  $c_{\nu}^{\pm}$  are constants which will be chosen below. By analytic continuation,  $\psi_{\pm}(\cdot, \omega)$  are solutions to Eq. (4.16). Furthermore, from the standard Hankel asymptotics

$$H_{\nu}^{\pm}(z) = \sqrt{\frac{2}{\pi z}} e^{\pm i(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)} [1 + O_{\mathbb{C}}(z^{-1})]$$

---

<sup>2</sup>We assume here for simplicity that  $d$  is odd, i.e.,  $\nu = \frac{d}{2} + \ell - 1$  is not an integer. In the case  $d$  even one may encounter a logarithmic loss but this does not affect the final result.



we see that  $c_\nu^\pm$  can be chosen in such a way that

$$\psi_\pm(r, \omega) = \frac{1}{\sqrt{2}} \mu^{-\frac{1}{4}} e^{\pm \mu^{1/2} r} [1 + O_{\mathbb{C}}(r^{-1} \omega^{-\frac{1}{2}})]$$

provided  $|\mu^{\frac{1}{2}} r| \geq 1$  and  $\omega \gg 1$ . It follows that  $W(\psi_-(\cdot, \omega), \psi_+(\cdot, \omega)) = 1$  and thus,  $\{\psi_\pm(\cdot, \omega)\}$  is a fundamental system for Eq. (4.16). Consequently, we intend to construct a solution of the integral equation

$$\begin{aligned} \tilde{v}_-(r, \omega) &= \psi_-(r, \omega) + \psi_-(r, \omega) \int_r^1 \psi_+(s, \omega) O_{\mathbb{C}}(\langle s \rangle^2) \tilde{v}_-(s, \omega) ds \\ &\quad - \psi_+(r, \omega) \int_r^1 \psi_-(s, \omega) O_{\mathbb{C}}(\langle s \rangle^2) \tilde{v}_-(s, \omega) ds. \end{aligned}$$

The functions  $\psi_\pm(\cdot, \omega)$  do not have zeros on  $[c\omega^{-\frac{1}{2}}, \infty)$  if  $c \geq 1$  is sufficiently large and thus, we may set  $\tilde{v}_- = \psi_- h_-$  and derive the Volterra equation

$$h_-(r, \omega) = 1 + \int_r^1 K(r, s, \omega) h_-(s, \omega) ds$$

for the function  $h_-$  where

$$K(r, s, \omega) = \left[ \psi_-(s, \omega) \psi_+(s, \omega) - \frac{\psi_+(r, \omega)}{\psi_-(r, \omega)} \psi_-(s, \omega)^2 \right] O_{\mathbb{C}}(\langle s \rangle^2).$$

We derive the bound

$$|K(r, s, \omega)| \lesssim |\mu|^{-\frac{1}{2}} + \left| \mu^{-\frac{1}{2}} e^{-2\mu^{1/2}(s-r)} \right| \lesssim \omega^{-\frac{1}{2}}$$

provided  $c\omega^{-\frac{1}{2}} \leq r \leq s \leq 1$  and thus,

$$\int_{c\omega^{-1/2}}^1 \sup_{r \in [c\omega^{-1/2}, s]} |K(r, s, \omega)| ds \lesssim \omega^{-\frac{1}{2}}.$$

Consequently, a standard Volterra iteration yields a solution  $h_-$  with the bound  $|h_-(r, \omega) - 1| \lesssim \omega^{-\frac{1}{2}}$  for  $r \in [c\omega^{-\frac{1}{2}}, 1]$ . Dividing by  $c_\nu^-$  yields the stated form of  $\tilde{v}_-$ . The solution  $\tilde{v}_+$  follows from the reduction ansatz, cf. the proof of Proposition 4.3.  $\square$

Thanks to the global representation  $H_\nu^\pm = J_\nu \pm i Y_\nu$  it is easy to glue together the fundamental systems from Lemmas 4.5 and 4.6.

**Corollary 4.7.** *Let  $\nu_0 > 0$ . Then we have the representations*

$$\begin{aligned} \tilde{v}_\pm(r, \omega) &= O_{\mathbb{C}}(\omega^0) v_0(r, \omega) \mp [i + O_{\mathbb{C}}(\omega^{-\frac{1}{2}})] v_1(r, \omega) \\ v_0(r, \omega) &= [\tilde{\beta}_- + O_{\mathbb{C}}(\omega^{-\frac{1}{2}})] \tilde{v}_-(r, \omega) + [\tilde{\beta}_+ + O_{\mathbb{C}}(\omega^{-\frac{1}{2}})] \tilde{v}_+(r, \omega) \end{aligned}$$

for all  $r \in (0, 1]$ ,  $\omega \gg 1$ , and  $\nu \in [0, \nu_0]$  where  $\tilde{\beta}_\pm \in \mathbb{C} \setminus \{0\}$ .

*Proof.* Since  $\{v_0, v_1\}$  is a fundamental system for Eq. (4.15) and  $\tilde{v}_-$  is a solution to that equation, there must exist connection coefficients  $\alpha_j(\omega)$  such that

$$\tilde{v}_-(r, \omega) = \alpha_0(\omega)v_0(r, \omega) + \alpha_1(\omega)v_1(r, \omega).$$

The coefficients are given by the Wronskian expressions

$$\alpha_0(\omega) = \frac{W(\tilde{v}_-(\cdot, \omega), v_1(\cdot, \omega))}{W(v_0(\cdot, \omega), v_1(\cdot, \omega))}, \quad \alpha_1(\omega) = \frac{W(\tilde{v}_-(\cdot, \omega), v_0(\cdot, \omega))}{W(v_1(\cdot, \omega), v_0(\cdot, \omega))}.$$

For the following computations it is useful to recall the formulae

$$\begin{aligned} W(pf, qg) &= W(p, q)fg + pqW(f, g) \\ W(f \circ \varphi, g \circ \varphi) &= [W(f, g) \circ \varphi]\varphi'. \end{aligned}$$

We have

$$\begin{aligned} W(v_0(\cdot, \omega), v_1(\cdot, \omega)) &= i\mu^{\frac{1}{2}}rW(J_\nu, Y_\nu)(i\mu^{\frac{1}{2}}r)[1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}})] \\ &= \frac{2}{\pi} + O_{\mathbb{C}}(\omega^{-\frac{1}{2}}). \end{aligned}$$

Furthermore, by evaluating the Wronskians at  $r = c\omega^{-\frac{1}{2}}$  we find

$$\begin{aligned} W(\tilde{v}_-(\cdot, \omega), v_1(\cdot, \omega)) &= i\mu^{\frac{1}{2}}rW(H_\nu^+, Y_\nu + O_{\mathbb{C}}(\omega^0)J_\nu)(i\mu^{\frac{1}{2}}r) + O_{\mathbb{C}}(\omega^{-\frac{1}{2}}) \\ &= O_{\mathbb{C}}(\omega^0) \end{aligned}$$

and

$$\begin{aligned} W(\tilde{v}_-(\cdot, \omega), v_0(\cdot, \omega)) &= i\mu^{\frac{1}{2}}rW(H_\nu^+, J_\nu)(i\mu^{\frac{1}{2}}r) + O_{\mathbb{C}}(\omega^{-\frac{1}{2}}) \\ &= i\mu^{\frac{1}{2}}rW(J_\nu + iY_\nu, J_\nu)(i\mu^{\frac{1}{2}}r) + O_{\mathbb{C}}(\omega^{-\frac{1}{2}}) \\ &= -\frac{2i}{\pi} + O_{\mathbb{C}}(\omega^{-\frac{1}{2}}). \end{aligned}$$

Consequently, we infer  $\alpha_0(\omega) = O_{\mathbb{C}}(\omega^0)$  and  $\alpha_1(\omega) = i + O_{\mathbb{C}}(\omega^{-\frac{1}{2}})$  as claimed. The proof for  $\tilde{v}_+$  is analogous. For the representation of  $v_0$  we use

$$v_0(r, \omega) = \frac{W(v_0(\cdot, \omega), \tilde{v}_+(\cdot, \omega))}{W(\tilde{v}_-(\cdot, \omega), \tilde{v}_+(\cdot, \omega))}\tilde{v}_-(r, \omega) + \frac{W(v_0(\cdot, \omega), \tilde{v}_-(\cdot, \omega))}{W(\tilde{v}_+(\cdot, \omega), \tilde{v}_-(\cdot, \omega))}\tilde{v}_+(r, \omega).$$

□

#### 4.4. A global fundamental system for small angular momenta.

In order to obtain a global fundamental system, it suffices to derive a representation of  $v_-$  in terms of the basis  $\{\tilde{v}_-, \tilde{v}_+\}$ .

**Lemma 4.8.** *Let  $\nu_0 > 0$  and  $\mu = b + i\omega$  where  $b \in \mathbb{R}$  is fixed. Then we have the representation*

$$v_-(r, \omega) = \tilde{\alpha}_- e^{\mu^{1/2}}[1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}})]\tilde{v}_-(r, \omega) + e^{-\mu^{1/2}}O_{\mathbb{C}}(\omega^{-\frac{1}{2}})\tilde{v}_+(r, \omega)$$

for all  $r \geq c\omega^{-\frac{1}{2}}$ ,  $\omega \gg c^2$ , and  $\nu \in [0, \nu_0]$ , where  $\tilde{\alpha}_- \in \mathbb{C} \setminus \{0\}$  and  $c \geq 1$  is sufficiently large.

*Proof.* We have

$$v_-(r, \omega) = \tilde{\alpha}_-(\omega)\tilde{v}_-(r, \omega) + \tilde{\alpha}_+(\omega)\tilde{v}_+(r, \omega)$$

where

$$\tilde{\alpha}_-(\omega) = \frac{W(v_-(\cdot, \omega), \tilde{v}_+(\cdot, \omega))}{W(\tilde{v}_-(\cdot, \omega), \tilde{v}_+(\cdot, \omega))}, \quad \tilde{\alpha}_+(\omega) = \frac{W(v_-(\cdot, \omega), \tilde{v}_-(\cdot, \omega))}{W(\tilde{v}_+(\cdot, \omega), \tilde{v}_-(\cdot, \omega))}.$$

We will calculate these Wronskians by evaluation at  $r = 1$ . From Proposition 4.3 and  $\xi(\mu^{-\frac{1}{2}}) = 0$ , see Eq. (4.10), we infer

$$\begin{aligned} v_-(1, \omega) &= \frac{1}{\sqrt{2}}\mu^{-\frac{1}{4}}[1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}})] \\ v'_-(1, \omega) &= -\frac{1}{\sqrt{2}}\mu^{\frac{1}{4}}[1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}})]. \end{aligned}$$

Furthermore, the Hankel asymptotics imply

$$\begin{aligned} \tilde{v}_{\pm}(1, \omega) &= c_{\nu}^{\pm}\mu^{-\frac{1}{4}}e^{\pm\mu^{1/2}}[1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}})] \\ \tilde{v}'_{\pm}(1, \omega) &= \pm c_{\nu}^{\pm}\mu^{\frac{1}{4}}e^{\pm\mu^{1/2}}[1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}})] \end{aligned}$$

for suitable constants  $c_{\nu}^{\pm} \in \mathbb{C} \setminus \{0\}$ . This yields

$$W(\tilde{v}_-(\cdot, \omega), \tilde{v}_+(\cdot, \omega)) = 2c_{\nu}^{-}c_{\nu}^{+} + O_{\mathbb{C}}(\omega^{-\frac{1}{2}})$$

and

$$\begin{aligned} W(v_-(\cdot, \omega), \tilde{v}_+(\cdot, \omega)) &= \sqrt{2}c_{\nu}^{+}e^{\mu^{1/2}}[1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}})] \\ W(v_-(\cdot, \omega), \tilde{v}_-(\cdot, \omega)) &= e^{-\mu^{1/2}}O_{\mathbb{C}}(\omega^{-\frac{1}{2}}) \end{aligned}$$

which implies the claim.  $\square$

It is now a simple matter to obtain global representations for  $v_0$  and  $v_-$ .

**Lemma 4.9.** *Let  $\nu_0 > 0$  and  $\mu = b + i\omega$  where  $b \in \mathbb{R}$  is fixed. Then we have the representations*

$$\begin{aligned} v_-(r, \omega) &= e^{\mu^{1/2}}O_{\mathbb{C}}(\omega^0)v_0(r, \omega) + \alpha_1e^{\mu^{1/2}}[1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}})]v_1(r, \omega) \\ v_0(r, \omega) &= e^{\mu^{1/2}}O_{\mathbb{C}}(\omega^{-\frac{1}{2}})v_-(r, \omega) + \alpha_+e^{\mu^{1/2}}[1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}})]v_+(r, \omega) \end{aligned}$$

for all  $r > 0$ ,  $\omega \gg 1$ , and  $\nu \in [0, \nu_0]$ , where  $\alpha_1, \alpha_+ \in \mathbb{C} \setminus \{0\}$ .

*Proof.* The claimed representation of  $v_-$  is a consequence of Corollary 4.7 and Lemma 4.8. For the representation of  $v_0$  we note that

$$\tilde{v}_{\pm}(r, \omega) = \frac{W(\tilde{v}_{\pm}(\cdot, \omega), v_+(\cdot, \omega))}{W(v_-(\cdot, \omega), v_+(\cdot, \omega))}v_-(r, \omega) + \frac{W(\tilde{v}_{\pm}(\cdot, \omega), v_-(\cdot, \omega))}{W(v_+(\cdot, \omega), v_-(\cdot, \omega))}v_+(r, \omega)$$

and from Proposition 4.3 we infer

$$W(v_-(\cdot, \omega), v_+(\cdot, \omega)) = 1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}}).$$

Furthermore, as in the proof of Lemma 4.8 we obtain

$$\begin{aligned} W(v_-(\cdot, \omega), \tilde{v}_+(\cdot, \omega)) &= \sqrt{2}c_{\nu}^+ e^{\mu^{1/2}} [1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}})] \\ W(v_-(\cdot, \omega), \tilde{v}_-(\cdot, \omega)) &= e^{-\mu^{1/2}} O_{\mathbb{C}}(\omega^{-\frac{1}{2}}) \\ W(\tilde{v}_-(\cdot, \omega), v_+(\cdot, \omega)) &= \sqrt{2}c_{\nu}^- e^{-\mu^{1/2}} [1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}})] \\ W(\tilde{v}_+(\cdot, \omega), v_+(\cdot, \omega)) &= e^{\mu^{1/2}} O_{\mathbb{C}}(\omega^{-\frac{1}{2}}) \end{aligned}$$

and the claim follows from Corollary 4.7.  $\square$

**4.5. A fundamental system near the center for large angular momenta.** The fundamental systems constructed in Section 4.3 are not useful as  $\nu \rightarrow \infty$  since the error terms are not controlled in this limit. Thus, we need yet another construction which covers the case of large angular momenta. We rewrite Eq. (4.4) as<sup>3</sup>

$$v''(r) - \frac{\nu^2}{r^2}v(r) - \mu v(r) = -\frac{1}{4r^2}v(r) + O_{\mathbb{C}}(\langle r \rangle^2)v(r) \quad (4.18)$$

and consider the “homogeneous” version

$$v''(r) - \frac{\nu^2}{r^2}v(r) - \mu v(r) = 0. \quad (4.19)$$

We rescale by introducing  $v(r) = w(\nu^{-1}\mu^{\frac{1}{2}}r)$  with  $\mu > 0$ ,  $\nu \geq 1$ , which yields

$$w''(y) - \nu^2(1 + \frac{1}{y^2})w(y) = 0$$

where  $y = \nu^{-1}\mu^{\frac{1}{2}}r$ . The Liouville-Green transform  $\tilde{w}(\zeta(y)) = |\zeta'(y)|^{\frac{1}{2}}w(y)$  leads to

$$\tilde{w}''(\zeta(y)) - \nu^2 \frac{1 + \frac{1}{y^2}}{\zeta'(y)^2} \tilde{w}(\zeta(y)) - \frac{q(y)}{\zeta'(y)^2} \tilde{w}(\zeta(y)) = 0$$

with

$$q(y) = \frac{1}{2} \frac{\zeta'''(y)}{\zeta'(y)} - \frac{3}{4} \frac{\zeta''(y)^2}{\zeta'(y)^2}.$$

Consequently, we would like to have

$$\zeta'(y) = \sqrt{1 + \frac{1}{y^2}}$$

and

$$\zeta(y) = \sqrt{1 + y^2} + \log \frac{y}{1 + \sqrt{1 + y^2}} + \gamma \quad (4.20)$$

---

<sup>3</sup>It is a well known “trick” from the asymptotic theory of Bessel functions to leave the singular term  $-\frac{1}{4r^2}v(r)$  on the right-hand side, see [22].

does the job, where  $\gamma \in \mathbb{C}$  is a free constant which we will choose in a moment. With this choice of  $\zeta$  we obtain

$$q(y) = \frac{1 + 6y^2}{4y^2(1 + y^2)^2}$$

and the equation

$$w''(y) - \nu^2(1 + \frac{1}{y^2})w(y) + q(y)w(y) = 0$$

has the fundamental system  $\zeta'(y)^{-\frac{1}{2}}e^{\pm\nu\zeta(y)}$ . Consequently, the equation

$$v''(r) - \frac{\nu^2}{r^2}v(r) + \alpha^2q(\alpha r)v(r) - \mu v(r) = 0, \quad \alpha = \nu^{-1}\mu^{\frac{1}{2}} \quad (4.21)$$

has the fundamental system  $\zeta'(\alpha r)^{-\frac{1}{2}}e^{\pm\nu\zeta(\alpha r)}$  and by choosing  $\gamma$  in Eq. (4.20) accordingly, we may normalize such that<sup>4</sup>  $\zeta(\alpha) = 0$ . This suggests to rewrite Eq. (4.18) as

$$v''(r) - \frac{\nu^2}{r^2}v(r) + \alpha^2q(\alpha r)v(r) - \mu v(r) = \alpha^2\tilde{q}(\alpha r)v(r) + O_{\mathbb{C}}(\langle r \rangle^2)v(r) \quad (4.22)$$

with

$$\alpha^2\tilde{q}(\alpha r) = \alpha^2q(\alpha r) - \frac{1}{4r^2} = \alpha^2 \frac{4 - \alpha^2r^2}{4(1 + \alpha^2r^2)^2}.$$

We emphasize that the function  $\alpha^2\tilde{q}(\alpha r)$  is regular at  $r = 0$  which is crucial for the following. This is the reason why one has to leave the term  $-\frac{1}{4r^2}v(r)$  on the right-hand side of Eq. (4.18).

**4.5.1. Analysis of  $\zeta$  and  $\tilde{q}$ .** As before, we need the analytic continuations of  $\zeta(\alpha r)$  and  $\alpha^2\tilde{q}(\alpha r)$  for  $\alpha = \nu^{-1}\mu^{\frac{1}{2}}$  with  $\mu = b + i\omega$ . The analytic continuation of  $\tilde{q}$  is manifest since it is a rational function. Furthermore, the arguments of the square root and the logarithm in  $\zeta(\alpha r)$  stay in  $\mathbb{C} \setminus (-\infty, 0]$  for all  $r \geq 0$ ,  $\omega \gg 1$ , and  $\nu \geq 1$ . Consequently, the desired analytic continuation is obtained by using principal branches.

**Lemma 4.10.** *Let  $\mu = b + i\omega$  and  $\alpha = \nu^{-1}\mu^{\frac{1}{2}}$  where  $b \in \mathbb{R}$  is fixed. Then we have the bound*

$$|\alpha^2\tilde{q}(\alpha r)| \lesssim |\alpha|^2\langle \alpha r \rangle^{-2}$$

for all  $r \geq 0$ ,  $\omega \gg 1$ , and  $\nu \geq 1$ .

---

<sup>4</sup>Again, one should write  $\zeta(\alpha r; \alpha)$  but for brevity we use the sloppier  $\zeta(\alpha r)$ .

*Proof.* The statement follows from

$$\begin{aligned} |1 + \mu x|^2 &= 1 + 2\operatorname{Re} \mu x + |\mu|^2 x^2 = 1 + \frac{2b}{|\mu|} |\mu| x + |\mu|^2 x^2 \\ &\geq 1 - \frac{2b^2}{b^2 + \omega^2} + \frac{1}{2} |\mu|^2 x^2 \\ &\geq \frac{1}{2} (1 + |\mu|^2 x^2) \end{aligned}$$

which is true for all  $x \in \mathbb{R}$  and  $\omega \gg 1$ .  $\square$

The only information on  $\zeta$  we are going to use is the following monotonicity property.

**Lemma 4.11.** *Let  $\mu = b + i\omega$  and  $\alpha = \nu^{-1}\mu^{\frac{1}{2}}$  where  $b \geq 0$  is fixed. Then we have*

$$\partial_r \operatorname{Re} \zeta(\alpha r) \geq 0$$

for all  $r > 0$ ,  $\omega \gg 1$ , and  $\nu \geq 1$ .

*Proof.* We have

$$\begin{aligned} \operatorname{Re} \zeta(\alpha r) &= \operatorname{Re} \sqrt{1 + \alpha^2 r^2} + \operatorname{Re} \log \frac{\alpha r}{1 + \sqrt{1 + \alpha^2 r^2}} + c_\alpha \\ &= \frac{1}{\sqrt{2}} \sqrt{|1 + \alpha^2 r^2| + 1 + \nu^{-2} b r^2} + \operatorname{Re} \log \frac{\alpha r}{1 + \sqrt{1 + \alpha^2 r^2}} + c_\alpha \end{aligned}$$

where  $c_\alpha$  is independent of  $r$ . Since  $|1 + \alpha^2 r^2|^2 = 1 + 2\nu^{-2} b r^2 + |\alpha|^4 r^4$  and  $b \geq 0$ , it is evident that the square root is monotonically increasing. Thus, it suffices to consider the logarithm. We have

$$\begin{aligned} \partial_r \operatorname{Re} \log \frac{\alpha r}{1 + \sqrt{1 + \alpha^2 r^2}} &= \operatorname{Re} \partial_r \log \frac{\alpha r}{1 + \sqrt{1 + \alpha^2 r^2}} \\ &= \operatorname{Re} \frac{1}{r \sqrt{1 + \alpha^2 r^2}} \\ &\geq 0. \end{aligned}$$

$\square$

4.5.2. *A fundamental system for Eq. (4.22).* The homogeneous equation (4.21) has the fundamental system

$$\frac{1}{\sqrt{2}} \mu^{-\frac{1}{4}} \zeta'(\alpha r)^{-\frac{1}{2}} e^{\pm \nu \zeta(\alpha r)}, \quad \alpha = \nu^{-1} \mu^{\frac{1}{2}}.$$

Now we construct a perturbative fundamental system for Eq. (4.22).

**Lemma 4.12.** *Let  $\mu = b + i\omega$  and  $\alpha = \nu^{-1}\mu^{\frac{1}{2}}$  where  $b \geq 0$  is fixed. Then Eq. (4.22) has a fundamental system  $\{\hat{v}_0, \hat{v}_1\}$  of the form*

$$\begin{aligned} \hat{v}_0(r, \omega) &= \frac{1}{\sqrt{2}} \mu^{-\frac{1}{4}} \zeta'(\alpha r)^{-\frac{1}{2}} e^{\nu \zeta(\alpha r)} [1 + O_{\mathbb{C}}(r^0 \omega^{-\frac{1}{2}} + \nu^{-1})] \\ \hat{v}_1(r, \omega) &= \frac{1}{\sqrt{2}} \mu^{-\frac{1}{4}} \zeta'(\alpha r)^{-\frac{1}{2}} e^{-\nu \zeta(\alpha r)} [1 + O_{\mathbb{C}}(r^0 \omega^{-\frac{1}{2}} + \nu^{-1})] \end{aligned}$$

for all  $r \in (0, 1]$ ,  $\omega \gg 1$ , and  $\nu \geq 1$ .

*Proof.* We set

$$\psi_{\pm}(r, \omega) := \frac{1}{\sqrt{2}} \mu^{-\frac{1}{4}} \zeta'(\alpha r)^{-\frac{1}{2}} e^{\pm \nu \zeta(\alpha r)}$$

and note that

$$\partial_r \psi_{\pm}(r, \omega) = \frac{1}{\sqrt{2}} \mu^{-\frac{1}{4}} \left[ \partial_r [\zeta'(\alpha r)^{-\frac{1}{2}}] \pm \nu \alpha \zeta'(\alpha r)^{\frac{1}{2}} \right] e^{\pm \nu \zeta(\alpha r)}$$

which yields  $W(\psi_{-}(\cdot, \omega), \psi_{+}(\cdot, \omega)) = 1$  since  $\nu \alpha = \mu^{\frac{1}{2}}$ . Consequently, our goal is to solve the integral equation

$$\begin{aligned} \hat{v}_0(r, \omega) &= \psi_{+}(r, \omega) \\ &+ \psi_{+}(r, \omega) \int_0^r \psi_{-}(s, \omega) [\alpha^2 \tilde{q}(\alpha s) + O_{\mathbb{C}}(\langle s \rangle^2)] \hat{v}_0(s, \omega) ds \\ &- \psi_{-}(r, \omega) \int_0^r \psi_{+}(s, \omega) [\alpha^2 \tilde{q}(\alpha s) + O_{\mathbb{C}}(\langle s \rangle^2)] \hat{v}_0(s, \omega) ds. \end{aligned}$$

The function  $\psi_{+}(\cdot, \omega)$  does not have zeros on  $(0, \infty)$  and hence, we may set  $h_0(r, \omega) := \frac{\hat{v}_0(r, \omega)}{\psi_{+}(r, \omega)}$  which leads to the Volterra equation

$$h_0(r, \omega) = 1 + \int_0^r K(r, s, \omega) h_0(s, \omega) ds$$

for  $h_0$  with the kernel

$$\begin{aligned} K(r, s, \omega) &= \left[ \psi_{-}(s, \omega) \psi_{+}(s, \omega) - \frac{\psi_{-}(r, \omega)}{\psi_{+}(r, \omega)} \psi_{+}(s, \omega)^2 \right] \\ &\times [\alpha^2 \tilde{q}(\alpha s) + O_{\mathbb{C}}(\langle s \rangle^2)]. \end{aligned}$$

Now we use Lemma 4.11 and  $|\zeta(\alpha r)|^{-\frac{1}{2}} \lesssim 1$  to obtain

$$\begin{aligned} |K(r, s, \omega)| &\lesssim \omega^{-\frac{1}{2}} [1 + e^{-2\nu[\operatorname{Re} \zeta(\alpha r) - \operatorname{Re} \zeta(\alpha s)]}] [|\alpha^2 \tilde{q}(\alpha s)| + \langle s \rangle^{-2}] \\ &\lesssim \omega^{-\frac{1}{2}} |\alpha|^2 \langle \alpha s \rangle^{-2} + \omega^{-\frac{1}{2}} \langle s \rangle^{-2} \end{aligned}$$

for all  $0 < s \leq r$  where the last estimate follows from Lemma 4.10. Consequently, we infer

$$\begin{aligned} \int_0^1 \sup_{r \in (s, 1)} |K(r, s, \omega)| ds &\lesssim \omega^{-\frac{1}{2}} |\alpha|^2 \int_0^1 \langle \alpha s \rangle^{-2} ds + \omega^{-\frac{1}{2}} \int_0^1 \langle s \rangle^{-2} ds \\ &\lesssim \omega^{-\frac{1}{2}} |\alpha| \int_0^{|\alpha|} \langle s \rangle^{-2} ds + \omega^{-\frac{1}{2}} \\ &\lesssim \nu^{-1} + \omega^{-\frac{1}{2}} \end{aligned}$$

since  $\omega^{-\frac{1}{2}} |\alpha| \simeq \nu^{-1}$  and a Volterra iteration yields the existence of  $h_0$  with  $|h_0(r, \omega)| \lesssim 1$  for all  $r \in [0, 1]$ ,  $\omega \gg 1$ , and  $\nu \geq 1$ . By re-inserting

this estimate into the Volterra equation for  $h_0$ , we find the desired bound  $|h_0(r, \omega) - 1| \lesssim \nu^{-1} + \omega^{-\frac{1}{2}}$ . The solution  $\hat{v}_1$  is constructed by using the reduction ansatz.  $\square$

#### 4.6. A global fundamental system for large angular momenta.

Now we glue together the fundamental systems from Proposition 4.3 and Lemma 4.12 in order to obtain a global fundamental system for large  $\nu$ . This time there is no need for an intermediate regime since the corresponding Wronskians can be evaluated at  $r = 1$ .

**Lemma 4.13.** *Let  $\mu = b + i\omega$  where  $b \geq 0$  is fixed. Then we have the representations*

$$v_-(r, \omega) = O_{\mathbb{C}}(\omega^{-\frac{1}{2}} + \nu^{-1})\hat{v}_0(r, \omega) + [1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}} + \nu^{-1})]\hat{v}_1(r, \omega)$$

$$\hat{v}_0(r, \omega) = O_{\mathbb{C}}(\omega^{-\frac{1}{2}} + \nu^{-1})v_-(r, \omega) + [1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}} + \nu^{-1})]v_+(r, \omega)$$

for all  $r > 0$ ,  $\omega \gg 1$ , and  $\nu \geq 1$ .

*Proof.* We have

$$v_-(r, \omega) = \frac{W(v_-(\cdot, \omega), \hat{v}_1(\cdot, \omega))}{W(\hat{v}_0(\cdot, \omega), \hat{v}_1(\cdot, \omega))}\hat{v}_0(r, \omega) + \frac{W(v_-(\cdot, \omega), \hat{v}_0(\cdot, \omega))}{W(\hat{v}_1(\cdot, \omega), \hat{v}_0(\cdot, \omega))}\hat{v}_1(r, \omega)$$

and from Lemma 4.12 we obtain

$$W(\hat{v}_0(\cdot, \omega), \hat{v}_1(\cdot, \omega)) = -1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}} + \nu^{-1}).$$

Furthermore, since  $\zeta(\alpha) = 0$  with  $\alpha = \nu^{-1}\mu^{\frac{1}{2}}$ , we obtain from Lemma 4.12 the expressions

$$\hat{v}_0(1, \omega) = \frac{1}{\sqrt{2}}\mu^{-\frac{1}{4}}\zeta'(\alpha)^{-\frac{1}{2}}[1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}} + \nu^{-1})]$$

$$\hat{v}'_0(1, \omega) = \frac{1}{\sqrt{2}}\mu^{\frac{1}{4}}\zeta'(\alpha)^{\frac{1}{2}}[1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}} + \nu^{-1})]$$

$$\hat{v}_1(1, \omega) = \frac{1}{\sqrt{2}}\mu^{-\frac{1}{4}}\zeta'(\alpha)^{-\frac{1}{2}}[1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}} + \nu^{-1})]$$

$$\hat{v}'_1(1, \omega) = -\frac{1}{\sqrt{2}}\mu^{\frac{1}{4}}\zeta'(\alpha)^{\frac{1}{2}}[1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}} + \nu^{-1})]$$

and from Proposition 4.3 we infer

$$v_{\pm}(1, \omega) = \frac{1}{\sqrt{2}}\mu^{-\frac{1}{4}}\xi'(\mu^{-\frac{1}{2}})^{-\frac{1}{2}}[1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}}\nu^0)]$$

$$v'_{\pm}(1, \omega) = \pm \frac{1}{\sqrt{2}}\mu^{\frac{1}{4}}\xi'(\mu^{-\frac{1}{2}})^{\frac{1}{2}}[1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}}\nu^0)].$$

Now observe that

$$\begin{aligned} \frac{\xi'(\mu^{-\frac{1}{2}})^2}{\zeta'(\alpha)^2} &= \frac{1 + \frac{1}{\mu} + \frac{\nu^2}{\mu}}{1 + \frac{\nu^2}{\mu}} = 1 + \frac{1}{\mu + \nu^2} \\ &= 1 + O_{\mathbb{C}}(\omega^{-1}\nu^{-2}) \end{aligned}$$



and we infer

$$\begin{aligned} W(v_-(\cdot, \omega), \hat{v}_1(\cdot, \omega)) &= O_{\mathbb{C}}(\omega^{-\frac{1}{2}} + \nu^{-1}) \\ W(v_-(\cdot, \omega), \hat{v}_0(\cdot, \omega)) &= 1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}} + \nu^{-1}) \end{aligned}$$

which yields the claimed representation for  $v_-$ . The proof for  $\hat{v}_0$  is analogous.  $\square$

## 5. THE REDUCED RESOLVENT

We are now in a position to construct the reduced resolvent  $R_{\ell, m}(\lambda)$ , i.e., the solution operator to Eq. (4.1).

**5.1. Small angular momenta.** We start with the solution operator to Eq. (4.2) in the case  $\ell \leq \ell_0$  for some fixed  $\ell_0 > 0$ . In this case the functions  $v_-$  and  $v_0$  from Proposition 4.3 and Lemma 4.5, respectively, are relevant. By Lemma 4.9 we have

$$\begin{aligned} W(\omega) &:= W(v_-(\cdot, \omega), v_0(\cdot, \omega)) \\ &= \alpha_+ e^{\mu^{1/2}} [1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}})] W(v_-(\cdot, \omega), v_+(\cdot, \omega)) \\ &= \alpha_+ e^{\mu^{1/2}} [1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}})] \end{aligned}$$

with  $\alpha_+ \in \mathbb{C} \setminus \{0\}$  and  $\mu = b + i\omega$ ,  $b \in \mathbb{R}$  fixed (the last equality follows from the representation in Proposition 4.3). In particular, this implies that  $\{v_0, v_-\}$  is a fundamental system for the homogeneous version of Eq. (4.2) (provided  $\omega$  is sufficiently large). Furthermore,  $v_0(r, \omega)$  and  $v_-(r, \omega)$  are recessive as  $r \rightarrow 0+$  and  $r \rightarrow \infty$ , respectively. Thus, the variation of constants formula yields a solution  $v$  of Eq. (4.2) given by

$$\begin{aligned} v(r) &= \frac{v_0(r, \omega)}{W(\omega)} \int_r^\infty v_-(s, \omega) s^{\frac{d-1}{2}} e^{-s^2/2} f(s) ds \\ &\quad + \frac{v_-(r, \omega)}{W(\omega)} \int_0^r v_0(s, \omega) s^{\frac{d-1}{2}} e^{-s^2/2} f(s) ds. \end{aligned}$$

We define an operator  $\tilde{R}_\ell(\lambda)$  by

$$\tilde{R}_\ell(\lambda) \tilde{f}(r) := \int_0^\infty \tilde{G}_\ell(r, s, \omega) \tilde{f}(s) ds$$

where  $\lambda = d + b + i\omega$  and

$$\tilde{G}_\ell(r, s, \omega) := \frac{e^{\frac{1}{2}(r^2 - s^2)}}{W(\omega)} \begin{cases} v_0(r, \omega) v_-(s, \omega) & r \leq s \\ v_-(r, \omega) v_0(s, \omega) & r \geq s \end{cases}. \quad (5.1)$$

With this notation it follows that

$$v(r) = e^{-\frac{1}{2}r^2} \tilde{R}_\ell(\lambda) \left( |\cdot|^{\frac{d-1}{2}} f \right) (r)$$

and thus, the corresponding solution  $u$  to Eq. (4.1) is given by

$$u(r) = R_{\ell,m}(\lambda)f(r) = r^{-\frac{d-1}{2}}e^{\frac{1}{2}r^2}v(r) = r^{-\frac{d-1}{2}}\tilde{R}_\ell(\lambda)\left(|\cdot|^{\frac{d-1}{2}}f\right)(r).$$

From this equation it also follows that  $R_{\ell,m}$  is in fact independent of  $m$ . Our goal is to show that

$$\|R_{\ell,m}(\lambda)f\|_{L^2_{\text{rad}}(\mathbb{R}^d)} \leq C\|f\|_{L^2_{\text{rad}}(\mathbb{R}^d)}, \quad \lambda = d + b + i\omega$$

for all  $\omega \gg 1$  and  $\ell \leq \ell_0$ . By the above, this is equivalent to the bound

$$\|\tilde{R}_\ell(\lambda)\tilde{f}\|_{L^2(\mathbb{R}_+)} \leq C\|\tilde{f}\|_{L^2(\mathbb{R}_+)} \quad (5.2)$$

and a proof of (5.2) is the goal of this section.

**5.1.1. Kernel bounds.** The desired  $L^2$ -boundedness of  $\tilde{R}_\ell(\lambda)$  will be a consequence of the following estimate.

**Lemma 5.1.** *Let  $\ell_0 > 0$ . Then the kernel  $\tilde{G}_\ell$  defined in (5.1) satisfies the bound*

$$|\tilde{G}_\ell(r, s, \omega)| \lesssim \omega^{-\frac{1}{2}} \begin{cases} \langle \omega^{-\frac{1}{2}}r \rangle^{-\frac{1}{2}+\frac{b}{2}} \langle \omega^{-\frac{1}{2}}s \rangle^{-\frac{1}{2}-\frac{b}{2}} & r \leq s \\ \langle \omega^{-\frac{1}{2}}r \rangle^{-\frac{1}{2}-\frac{b}{2}} \langle \omega^{-\frac{1}{2}}s \rangle^{-\frac{1}{2}+\frac{b}{2}} & r \geq s \end{cases}$$

for all  $r, s > 0$ ,  $\omega \gg 1$ , and  $\ell \leq \ell_0$ .

*Proof.* By symmetry it suffices to consider  $r \geq s$ . Furthermore, we make frequent use of the estimate

$$|\xi'(\mu^{-\frac{1}{2}}r)|^{-1} \lesssim \langle \omega^{-\frac{1}{2}}r \rangle^{-1}$$

which is a consequence of Eq. (4.13). We distinguish different cases. In the following the constant  $c \geq 1$  is assumed to be so large that Lemma 4.6 holds.

- (1) Bessel-Bessel:  $0 < s \leq r \leq c\omega^{-\frac{1}{2}}$ . We use Lemma 4.9 to obtain the bound

$$|v_-(r, \omega)| \lesssim e^{\text{Re}(\mu^{1/2})} [|v_0(r, \omega)| + |v_1(r, \omega)|]$$

and Lemma 4.5 yields, with  $\nu = \frac{d}{2} + \ell - 1$ ,

$$\begin{aligned} |v_0(s, \omega)v_-(r, \omega)| &\lesssim e^{\text{Re}(\mu^{1/2})} s^{\frac{1}{2}} |\mu^{\frac{1}{2}}s|^\nu r^{\frac{1}{2}} \left( |\mu^{\frac{1}{2}}r|^\nu + |\mu^{\frac{1}{2}}r|^{-\nu} \right) \\ &\lesssim e^{\text{Re}(\mu^{1/2})} \omega^{-\frac{1}{2}} \left( |\omega^{\frac{1}{2}}r|^{2\nu} + 1 \right) \\ &\lesssim e^{\text{Re}(\mu^{1/2})} \omega^{-\frac{1}{2}}. \end{aligned}$$

This implies  $|\tilde{G}_\ell(r, s, \omega)| \lesssim \omega^{-\frac{1}{2}}$ .

- (2) Bessel-Hankel:  $0 < s \leq c\omega^{-\frac{1}{2}} \leq r \leq 1$ . From Lemmas 4.8, 4.6 and the Hankel asymptotics we infer

$$\begin{aligned} |v_-(r, \omega)| &\lesssim e^{\operatorname{Re}(\mu^{1/2})} \omega^{-\frac{1}{4}} e^{-\operatorname{Re}(\mu^{1/2})r} \\ &\quad + \omega^{-\frac{3}{4}} e^{-\operatorname{Re}(\mu^{1/2})} e^{\operatorname{Re}(\mu^{1/2})r} \\ &\lesssim \omega^{-\frac{1}{4}} e^{\operatorname{Re}(\mu^{1/2})} \end{aligned}$$

and thus,

$$|v_0(s, \omega)v_-(r, \omega)| \lesssim \omega^{-\frac{1}{2}} e^{\operatorname{Re}(\mu^{1/2})}$$

which yields the desired  $|\tilde{G}_\ell(r, s, \omega)| \lesssim \omega^{-\frac{1}{2}}$ .

- (3) Bessel-Weber:  $0 < s \leq c\omega^{-\frac{1}{2}} \leq 1 \leq r$ . Here we use Proposition 4.3 and Lemma 4.2 to obtain

$$\begin{aligned} |v_-(r, \omega)| &\lesssim \omega^{-\frac{1}{4}} \langle \omega^{-\frac{1}{2}} r \rangle^{-\frac{1}{2}} e^{-\operatorname{Re}[\mu \xi(\mu^{-1/2} r)]} \\ &\lesssim \omega^{-\frac{1}{4}} \langle \omega^{-\frac{1}{2}} r \rangle^{-\frac{1}{2} - \frac{b}{2}} e^{-\frac{1}{2} r^2} e^{-\varphi(r; \omega, \nu)} \\ &\lesssim \omega^{-\frac{1}{4}} \langle \omega^{-\frac{1}{2}} r \rangle^{-\frac{1}{2} - \frac{b}{2}} e^{-\frac{1}{2} r^2} \end{aligned}$$

since  $\varphi(\cdot; \omega, \nu)$  is monotonically increasing and  $|\varphi(1; \omega, \nu)| \lesssim 1$ . Consequently, with  $|v_0(s, \omega)| \lesssim \omega^{-\frac{1}{4}}$  we infer

$$|v_0(s, \omega)v_-(r, \omega)| \lesssim \omega^{-\frac{1}{2}} \langle \omega^{-\frac{1}{2}} r \rangle^{-\frac{1}{2} - \frac{b}{2}} e^{-\frac{1}{2} r^2}$$

which implies  $|\tilde{G}_\ell(r, s, \omega)| \lesssim \omega^{-\frac{1}{2}} \langle \omega^{-\frac{1}{2}} r \rangle^{-\frac{1}{2} - \frac{b}{2}}$ .

- (4) Hankel-Hankel:  $c\omega^{-\frac{1}{2}} \leq s \leq r \leq 1$ . From Corollary 4.7 and the Hankel asymptotics we have

$$\begin{aligned} |v_0(s, \omega)| &\lesssim |\tilde{v}_-(s, \omega)| + |\tilde{v}_+(s, \omega)| \\ &\lesssim \omega^{-\frac{1}{4}} e^{\operatorname{Re}(\mu^{1/2})s} \end{aligned}$$

and from the Bessel-Hankel-case we recall the estimate

$$|v_-(r, \omega)| \lesssim \omega^{-\frac{1}{4}} e^{\operatorname{Re}(\mu^{1/2})} e^{-\operatorname{Re}(\mu^{1/2})r} + \omega^{-\frac{3}{4}}.$$

This yields

$$\begin{aligned} |v_0(s, \omega)v_-(r, \omega)| &\lesssim \omega^{-\frac{1}{2}} e^{\operatorname{Re}(\mu^{1/2})} e^{-\operatorname{Re}(\mu^{1/2})(r-s)} \\ &\quad + \omega^{-1} e^{\operatorname{Re}(\mu^{1/2})s} \\ &\lesssim \omega^{-\frac{1}{2}} e^{\operatorname{Re}(\mu^{1/2})} \end{aligned}$$

since  $\operatorname{Re}(\mu^{1/2}) \geq 0$  and we obtain  $|\tilde{G}_\ell(r, s, \omega)| \lesssim \omega^{-\frac{1}{2}}$ .

- (5) Hankel-Weber:  $c\omega^{-\frac{1}{2}} \leq s \leq 1 \leq r$ . From Hankel-Hankel and Bessel-Weber we infer

$$|v_0(s, \omega)v_-(r, \omega)| \lesssim \omega^{-\frac{1}{2}} e^{\operatorname{Re}(\mu^{1/2})} \langle \omega^{-\frac{1}{2}} r \rangle^{-\frac{1}{2}-\frac{b}{2}} e^{-\frac{1}{2}r^2}$$

which yields  $|\tilde{G}_\ell(r, s, \omega)| \lesssim \omega^{-\frac{1}{2}} \langle \omega^{-\frac{1}{2}} r \rangle^{-\frac{1}{2}-\frac{b}{2}}$ .

- (6) Weber-Weber:  $1 \leq s \leq r$ . From Lemma 4.9 and Proposition 4.3 we infer

$$\begin{aligned} |v_0(s, \omega)| &\lesssim \omega^{-\frac{3}{4}} e^{\operatorname{Re}(\mu^{1/2})} \langle \omega^{-\frac{1}{2}} s \rangle^{-\frac{1}{2}-\frac{b}{2}} e^{-\frac{1}{2}s^2} e^{-\varphi(s; \omega, \nu)} \\ &\quad + \omega^{-\frac{1}{4}} e^{\operatorname{Re}(\mu^{1/2})} \langle \omega^{-\frac{1}{2}} s \rangle^{-\frac{1}{2}+\frac{b}{2}} e^{\frac{1}{2}s^2} e^{\varphi(s; \omega, \nu)} \end{aligned}$$

which yields

$$\begin{aligned} |v_0(s, \omega)v_-(r, \omega)| &\lesssim \omega^{-\frac{1}{2}} e^{\operatorname{Re}(\mu^{1/2})} \langle \omega^{-\frac{1}{2}} r \rangle^{-\frac{1}{2}-\frac{b}{2}} \langle \omega^{-\frac{1}{2}} s \rangle^{-\frac{1}{2}+\frac{b}{2}} \\ &\quad \times \left[ \omega^{-\frac{1}{2}} e^{-\frac{1}{2}r^2-\frac{1}{2}s^2} e^{-\varphi(s; \omega, \nu)} e^{-\varphi(r; \omega, \nu)} \right. \\ &\quad \left. + e^{-\frac{1}{2}(r^2-s^2)} e^{-[\varphi(r; \omega, \nu)-\varphi(s; \omega, \nu)]} \right] \\ &\lesssim \omega^{-\frac{1}{2}} e^{\operatorname{Re}(\mu^{1/2})} \langle \omega^{-\frac{1}{2}} r \rangle^{-\frac{1}{2}-\frac{b}{2}} \langle \omega^{-\frac{1}{2}} s \rangle^{-\frac{1}{2}+\frac{b}{2}} \\ &\quad \times e^{-\frac{1}{2}(r^2-s^2)} \end{aligned}$$

since  $\varphi(\cdot; \omega, \nu)$  is monotonically increasing and  $|\varphi(1; \omega, \nu)| \lesssim 1$ . Consequently, we infer

$$|\tilde{G}_\ell(r, s, \omega)| \lesssim \omega^{-\frac{1}{2}} \langle \omega^{-\frac{1}{2}} r \rangle^{-\frac{1}{2}-\frac{b}{2}} \langle \omega^{-\frac{1}{2}} s \rangle^{-\frac{1}{2}+\frac{b}{2}}.$$

□

**5.2. Large angular momenta.** For large angular momenta  $\ell \geq \ell_0$  we use the functions  $v_-$  and  $\hat{v}_0$  from Proposition 4.3 and Lemma 4.12, respectively, to construct the operator  $\tilde{R}_\ell(\lambda)$ . For the Wronskian of  $v_-$  and  $\hat{v}_0$  we obtain

$$\hat{W}(\omega) := W(v_-(\cdot, \omega), \hat{v}_0(\cdot, \omega)) = 1 + O_{\mathbb{C}}(\omega^{-\frac{1}{2}} + \nu^{-1}) \quad (5.3)$$

where we have used the global representation from Lemma 4.13 and  $W(v_-(\cdot, \omega), v_+(\cdot, \omega)) = 1$  which follows from Proposition 4.3. Eq. (5.3) shows that  $\{\hat{v}_0, v_-\}$  is a fundamental system for Eq. (4.3) provided  $\omega$  and  $\ell$  are sufficiently large (recall that  $\nu = \frac{d}{2} + \ell - 1$ ). Furthermore,  $\hat{v}_0(r, \omega)$  is recessive as  $r \rightarrow 0+$  and  $v_-(r, \omega)$  is recessive as  $r \rightarrow \infty$ . Thus, we set

$$\hat{G}_\ell(r, s, \omega) := \frac{e^{\frac{1}{2}(r^2-s^2)}}{\hat{W}(\omega)} \begin{cases} \hat{v}_0(r, \omega)v_-(s, \omega) & r \leq s \\ v_-(r, \omega)\hat{v}_0(s, \omega) & r \geq s \end{cases} \quad (5.4)$$

and obtain

$$\tilde{R}_\ell(\lambda)f(r) = \int_0^\infty \hat{G}_\ell(r, s, \omega)f(s)ds$$

in the case  $\ell \gg 1$ .

5.2.1. *Kernel bounds.* Now we show that  $\hat{G}_\ell$  satisfies the same bounds as  $\tilde{G}_\ell$ .

**Lemma 5.2.** *The kernel  $\hat{G}_\ell$  defined in (5.4) satisfies the bound*

$$|\hat{G}_\ell(r, s, \omega)| \lesssim \omega^{-\frac{1}{2}} \begin{cases} \langle \omega^{-\frac{1}{2}}r \rangle^{-\frac{1}{2}+\frac{b}{2}} \langle \omega^{-\frac{1}{2}}s \rangle^{-\frac{1}{2}-\frac{b}{2}} & r \leq s \\ \langle \omega^{-\frac{1}{2}}r \rangle^{-\frac{1}{2}-\frac{b}{2}} \langle \omega^{-\frac{1}{2}}s \rangle^{-\frac{1}{2}+\frac{b}{2}} & r \geq s \end{cases}$$

for all  $r, s > 0$ ,  $\omega \gg 1$ , and  $\ell \gg 1$ .

*Proof.* By symmetry it suffices to consider  $r \geq s$  and we distinguish three cases. As before we set  $\alpha = \nu^{-1}\mu^{\frac{1}{2}}$  and recall that  $\nu = \frac{d}{2} + \ell - 1$ . We will use the estimates  $|\zeta'(\alpha r)|^{-1} \lesssim 1$  and  $|\zeta'(\mu^{-\frac{1}{2}}r)|^{-1} \lesssim \langle \omega^{-\frac{1}{2}}r \rangle^{-1}$  which follow from Eq. (4.13).

(1) Bessel-Bessel:  $0 < s \leq r \leq 1$ . From Lemma 4.12 we obtain

$$|\hat{v}_0(s, \omega)| \lesssim \omega^{-\frac{1}{4}} e^{\nu \operatorname{Re} \zeta(\alpha s)}$$

and Lemma 4.13 yields

$$\begin{aligned} |v_-(r, \omega)| &\lesssim |\hat{v}_0(r, \omega)| + |\hat{v}_1(r, \omega)| \\ &\lesssim \omega^{-\frac{1}{4}} e^{\nu \operatorname{Re} \zeta(\alpha r)} + \omega^{-\frac{1}{4}} e^{-\nu \operatorname{Re} \zeta(\alpha r)}. \end{aligned}$$

Thus, we infer

$$\begin{aligned} |\hat{v}_0(s, \omega)v_-(r, \omega)| &\lesssim \omega^{-\frac{1}{2}} e^{\nu \operatorname{Re} \zeta(\alpha s) + \nu \operatorname{Re} \zeta(\alpha r)} \\ &\quad + \omega^{-\frac{1}{2}} e^{-\nu [\operatorname{Re} \zeta(\alpha r) - \operatorname{Re} \zeta(\alpha s)]} \\ &\lesssim \omega^{-\frac{1}{2}} e^{2\nu \operatorname{Re} \zeta(\alpha)} + \omega^{-\frac{1}{2}} \lesssim \omega^{-\frac{1}{2}} \end{aligned}$$

since  $r \mapsto \operatorname{Re} \zeta(\alpha r)$  is monotonically increasing by Lemma 4.11 and  $\zeta(\alpha) = 0$ . This yields  $|\hat{G}_\ell(r, s, \omega)| \lesssim \omega^{-\frac{1}{2}}$ .

(2) Bessel-Weber:  $0 < s \leq 1 \leq r$ . From Lemma 4.12 and Proposition 4.3 we infer

$$\begin{aligned} |\hat{v}_0(s, \omega)v_-(r, \omega)| &\lesssim \omega^{-\frac{1}{2}} \langle \omega^{-\frac{1}{2}}r \rangle^{-\frac{1}{2}-\frac{b}{2}} e^{-\frac{1}{2}r^2} e^{\nu \operatorname{Re} \zeta(\alpha s)} e^{-\varphi(r; \omega, \nu)} \\ &\lesssim \omega^{-\frac{1}{2}} \langle \omega^{-\frac{1}{2}}r \rangle^{-\frac{1}{2}-\frac{b}{2}} e^{-\frac{1}{2}r^2} \end{aligned}$$

where we have used the fact that  $s \mapsto \operatorname{Re} \zeta(\alpha s)$  and  $\varphi(\cdot; \omega, \nu)$  are monotonically increasing (Lemmas 4.11, 4.2) as well as  $\zeta(\alpha) = 0$  and  $|\varphi(1; \omega, \nu)| \lesssim 1$ . Consequently, we obtain the desired

$$|\hat{G}_\ell(r, s, \omega)| \lesssim \omega^{-\frac{1}{2}} \langle \omega^{-\frac{1}{2}}r \rangle^{-\frac{1}{2}-\frac{b}{2}}.$$

- (3) Weber-Weber:  $1 \leq s \leq r$ . Here we use the representation from Lemma 4.13, Proposition 4.3, and Lemma 4.2 to obtain

$$\begin{aligned} |\hat{v}_0(s, \omega)| &\lesssim |v_-(s, \omega)| + |v_+(s, \omega)| \\ &\lesssim \omega^{-\frac{1}{4}} \langle \omega^{-\frac{1}{2}} s \rangle^{-\frac{1}{2} - \frac{b}{2}} e^{-\frac{1}{2}s^2} e^{-\varphi(s; \omega, \nu)} \\ &\quad + \omega^{-\frac{1}{4}} \langle \omega^{-\frac{1}{2}} s \rangle^{-\frac{1}{2} + \frac{b}{2}} e^{\frac{1}{2}s^2} e^{\varphi(s; \omega, \nu)} \\ &\lesssim \omega^{-\frac{1}{4}} \langle \omega^{-\frac{1}{2}} s \rangle^{-\frac{1}{2} + \frac{b}{2}} e^{\frac{1}{2}s^2} [e^{-\varphi(1; \omega, \nu)} + e^{\varphi(s; \omega, \nu)}] \end{aligned}$$

and this implies

$$\begin{aligned} |\hat{v}_0(s, \omega) v_-(r, \omega)| &\lesssim \omega^{-\frac{1}{2}} \langle \omega^{-\frac{1}{2}} r \rangle^{-\frac{1}{2} - \frac{b}{2}} \langle \omega^{-\frac{1}{2}} s \rangle^{-\frac{1}{2} + \frac{b}{2}} e^{-\frac{1}{2}(r^2 - s^2)} \\ &\quad \times [e^{-2\varphi(1; \omega, \nu)} + e^{-[\varphi(r; \omega, \nu) - \varphi(s; \omega, \nu)]}] \\ &\lesssim \omega^{-\frac{1}{2}} \langle \omega^{-\frac{1}{2}} r \rangle^{-\frac{1}{2} - \frac{b}{2}} \langle \omega^{-\frac{1}{2}} s \rangle^{-\frac{1}{2} + \frac{b}{2}} e^{-\frac{1}{2}(r^2 - s^2)} \end{aligned}$$

since  $\varphi(\cdot; \omega, \nu)$  is monotonically increasing and  $|\varphi(1; \omega, \nu)| \lesssim 1$ .  $\square$

**5.3. Boundedness of the reduced resolvent.** We can now conclude the desired  $L^2$ -boundedness of  $\tilde{R}_\ell(\lambda)$  for all angular momenta. This concludes the proof of Theorem 2.6.

**Lemma 5.3.** *Let  $b > 0$ . Then we have the bound*

$$\|\tilde{R}_\ell(d + b + i\omega)\|_{L^2(\mathbb{R}_+)} \leq C$$

for all  $\omega \gg 1$  and all  $\ell \in \mathbb{N}_0$ .

*Proof.* We choose  $\ell_0 > 0$  so large that Lemma 5.2 applies for all  $\ell \geq \ell_0$ . For  $\ell \leq \ell_0$  we define an auxiliary operator  $T_\ell(\omega)$  by

$$T_\ell(\omega)f(r) := \int_0^\infty \omega^{\frac{1}{2}} \tilde{G}_\ell(\omega^{\frac{1}{2}} r, \omega^{\frac{1}{2}} s, \omega) f(s) ds.$$

From Lemma 5.1 we have the bound

$$|\omega^{\frac{1}{2}} \tilde{G}_\ell(\omega^{\frac{1}{2}} r, \omega^{\frac{1}{2}} s, \omega)| \lesssim \begin{cases} \langle r \rangle^{-\frac{1}{2} + \frac{b}{2}} \langle s \rangle^{-\frac{1}{2} - \frac{b}{2}} & r \leq s \\ \langle r \rangle^{-\frac{1}{2} - \frac{b}{2}} \langle s \rangle^{-\frac{1}{2} + \frac{b}{2}} & r \geq s \end{cases}$$

and this implies  $\|T_\ell(\omega)\|_{L^2(\mathbb{R}_+)} \leq C$  for all  $\omega \gg 1$  (see e.g. [4], Lemma 5.5). For  $a > 0$  we write  $f_a(r) := f(\frac{r}{a})$  and by scaling we obtain

$$\begin{aligned} \|\tilde{R}_\ell(\lambda)f\|_{L^2(\mathbb{R}_+)} &= \omega^{\frac{1}{4}} \|[\tilde{R}_\ell(\lambda)f]_{\omega^{-1/2}}\|_{L^2(\mathbb{R}_+)} \\ &= \omega^{\frac{1}{4}} \|T_\ell(\omega)f_{\omega^{-1/2}}\|_{L^2(\mathbb{R}_+)} \\ &\leq C \omega^{\frac{1}{4}} \|f_{\omega^{-1/2}}\|_{L^2(\mathbb{R}_+)} \\ &= C \|f\|_{L^2(\mathbb{R}_+)} \end{aligned}$$

where  $\lambda = d + b + i\omega$ . This proves the statement for all  $\ell \leq \ell_0$ . In the case  $\ell \geq \ell_0$  we replace  $\tilde{G}_\ell$  by  $\hat{G}_\ell$  and use Lemma 5.2.  $\square$

#### APPENDIX A. APPLICABILITY OF THE ABSTRACT THEORY

In this appendix we discuss the applicability of the abstract theory to deduce Theorem 1.1. The most general results related to spectral mapping are developed in [3]. To be more precise, the paper [3] deals with the following problem. Suppose we are given an “unperturbed semigroup”  $T_0(t)$  on a Banach space  $X$  with generator  $A$  and a “perturbed semigroup”  $T(t)$  with generator  $A + B$  where for our purposes  $B$  may be assumed bounded (the theory in [3] is more general). Under what assumptions on  $T_0$ ,  $B$ , and/or  $T$  does spectral mapping for  $T$  hold? The paper [3] derives various sufficient criteria based on norm continuity properties of the remainders in the Dyson-Phillips expansion. Unfortunately, many of the criteria involve the perturbed semigroup  $T$  itself or an infinite series of convolutions of the operators  $T_0$  and  $B$  which makes them hard to check. However, there is a set of criteria that involve the unperturbed semigroup  $T_0$  and the perturbing operator  $B$  only. Since there exists an explicit representation of the unperturbed Ornstein-Uhlenbeck semigroup  $S_0$ , one might hope to deduce Theorem 1.1 from the abstract theory. Let us recall the precise statement.

**Theorem A.1** (Brendle-Nagel-Poland [3]). *Let  $T_0$ ,  $A$ ,  $B$ , and  $T$  be as above. For  $k \in \mathbb{N}_0$  define  $\tilde{T}_k : [0, \infty) \rightarrow \mathcal{B}(X)$  recursively by*

$$\tilde{T}_k(t)f := \int_0^t \tilde{T}_{k-1}(t-s)BT_0(s)f ds, \quad \tilde{T}_0(t) := T_0(t).$$

*If there exists a  $k \in \mathbb{N}$  such that  $\tilde{T}_k$  is norm continuous on  $[0, \infty)$  then*

$$\sigma(T(t)) \setminus \{0\} \subset e^{t\sigma(A+B)} \cup \{\lambda : |\lambda| \leq r_{\text{crit}}(T_0(t))\}.$$

*Furthermore, if  $\tilde{T}_1$  is norm continuous on  $[0, \infty)$  then*

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A+B)} \cup \sigma_{\text{crit}}(T_0(t)) \setminus \{0\}.$$

The most accessible criterion is of course the case  $k = 1$ . In fact, this is also the property that is tested in Example 5.2 in [3]. In what follows we discuss this criterion for the Ornstein-Uhlenbeck operator  $L$ .

The free Ornstein-Uhlenbeck semigroup  $S_0(t)$  has the explicit representation

$$\begin{aligned} [S_0(t)f](x) &= \frac{1}{[\pi\alpha(t)]^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|y|^2}{\alpha(t)}} f(e^{-2t}x - y) dy \\ &=: K_t * f(e^{-2t}x) \end{aligned} \tag{A.1}$$

where  $\alpha(t) = 1 - e^{-4t}$ . Consequently,  $S_0(t)$  consists of a “heat part” and a “dilation part”. While the former is very well behaved, the latter is responsible for the difficulties of the problem at hand. Thus, it makes sense to start the discussion by considering the dilation semigroup alone. To this end, let  $[T_0(t)f](x) = f(e^{-2t}x)$  be a dilation semigroup on  $L^2(\mathbb{R}^d)$  and consider the perturbing operator  $Bf(x) = V(x)f(x)$  for some nonzero potential  $V$ , e.g.  $V \in C_c^\infty(\mathbb{R}^d)$ . We are interested in the operator

$$\tilde{T}_1(t)f = \int_0^t T_0(t-s)BT_0(s)f ds.$$

Explicitly, we have

$$\begin{aligned} [T_0(t-s)BT_0(s)f](x) &= V(e^{-2(t-s)}x)f(e^{-2t}x) \\ &= [T_0(t-s)V](x)[T_0(t)f](x) \end{aligned}$$

and this yields

$$\tilde{T}_1(t)f = T_0(t)f \int_0^t T_0(t-s)V ds$$

which is certainly *not* norm continuous for all  $t \geq 0$  since  $t \mapsto T_0(t)$  is not norm continuous at *any*  $t \geq 0$ . Similarly, one sees that the same is true for  $\tilde{T}_k$  for any  $k \in \mathbb{N}$ . Consequently, Theorem A.1 does not apply to perturbed dilation operators, not even if the perturbing potential is as “nice” as possible.

The situation for the Ornstein-Uhlenbeck semigroup is similar. In this case we have

$$\begin{aligned} [S_0(t-s)BS_0(s)f](x) &= \int_{\mathbb{R}^d} K_{t-s}(e^{-2(t-s)}x - y)V(y)[S_0(s)f](y)dy \\ &= \int_{\mathbb{R}^d} K_{t-s}(e^{-2(t-s)}x - y)V(y) \int_{\mathbb{R}^d} K_s(e^{-2s}y - y')f(y')dy'dy \\ &= e^{-2dt} \int_{\mathbb{R}^d} K_{t-s}(e^{-2(t-s)}(x - y))V(e^{-2(t-s)}y) \\ &\quad \times \int_{\mathbb{R}^d} K_s(e^{-2t}(y - y'))f(e^{-2t}y')dy' \end{aligned}$$

and thus, for  $\tilde{S}_1(t)f = \int_0^t S_0(t-s)BS_0(s)f ds$ , we find

$$\tilde{S}_1(t)f = e^{-2dt} \int_0^t [T_0(t-s)K_{t-s} * T_0(t-s)V][T_0(t)K_s * T_0(t)f]ds.$$

One may now estimate the convolutions using Young’s inequality. However, the presence of the term  $T_0(t)f$  leads to the same kind of norm



discontinuity we encountered in the above discussion of the dilation operator. Consequently, we do not see how to apply Theorem A.1 to the Ornstein-Uhlenbeck operator  $L$ . This justifies our approach via explicit construction of the reduced resolvent.

## APPENDIX B. ON THE SPECTRAL INCLUSION $\sigma(L_0) \subset \sigma(L)$

By a perturbative argument one can in fact show that the addition of the potential  $V$  does not at all alter the spectrum in the left half-plane  $\{z \in \mathbb{C} : \operatorname{Re} z \leq d\}$ . The point is that the structure of the spectrum of  $L_0$  is a consequence of the asymptotics of eigenfunctions as  $r \rightarrow \infty$ . The requirements on the potential, on the other hand, imply that the addition of the potential does not change the asymptotic behavior of eigenfunctions. Consequently, one has  $\sigma(L_0) \subset \sigma(L)$ . The precise argument is as follows.

**B.1. Proof of Lemma 2.2.** We use the adjoint operator

$$L_0^* u(x) = \Delta u(x) + 2x \cdot \nabla u(x) + 2du(x)$$

with domain given in [18] and  $L^* = L_0^* u + \bar{V}u$ . It suffices to show that

$$\{z \in \mathbb{C} : \operatorname{Re} z < d\} \subset \sigma_p(L^*).$$

In radial symmetry, the spectral equation  $(\lambda - L^*)u = 0$  reads

$$u''(r) + \frac{d-1}{r}u'(r) + 2ru'(r) - (\lambda - 2d)u(r) = -\bar{V}(r)u(r). \quad (\text{B.1})$$

Upon setting  $u(r) = r^{-\frac{d-1}{2}} e^{-r^2/2} v(r)$  we find the normal form equation

$$v''(r) - r^2 v(r) - (\lambda - d)v(r) = [O(r^{-2}) - \bar{V}(r)]v(r). \quad (\text{B.2})$$

The “homogeneous” equation  $v''(r) - r^2 v(r) - (\lambda - d)v(r) = 0$  has the fundamental system

$$U\left(\frac{\lambda-d}{2}, \sqrt{2}r\right), \quad V\left(\frac{\lambda-d}{2}, \sqrt{2}r\right)$$

where  $U$  and  $V$  are the standard parabolic cylinder functions (see [23]). We have the bounds

$$|U\left(\frac{\lambda-d}{2}, \sqrt{2}r\right)| \simeq e^{-r^2/2} r^{-\frac{\operatorname{Re} \lambda - d + 1}{2}}, \quad |V\left(\frac{\lambda-d}{2}, \sqrt{2}r\right)| \simeq e^{r^2/2} r^{\frac{\operatorname{Re} \lambda - d - 1}{2}}$$

for  $r \gg 1$ . Now one may easily set up a Volterra iteration, see Section 3.3, to treat the right-hand side in Eq. (B.2) perturbatively. This yields a fundamental system  $\{v_0, v_1\}$  of Eq. (B.2) which satisfies the same bounds as  $U(\frac{\lambda-d}{2}, \sqrt{2}r)$  and  $V(\frac{\lambda-d}{2}, \sqrt{2}r)$ . As a consequence, one infers a fundamental system  $\{u_0, u_1\}$  of Eq. (B.1) with

$$|u_0(r)| \simeq e^{-r^2} r^{-\frac{\operatorname{Re} \lambda}{2}}, \quad |u_1(r)| \simeq r^{\frac{\operatorname{Re} \lambda}{2} - d}$$

for  $r \gg 1$ . This shows that *any* solution of Eq. (B.1) belongs to  $L^2_{\text{rad}}(\mathbb{R}^d)$  near infinity provided  $\text{Re } \lambda < d$ . Thus, by taking the solution of Eq. (B.1) which is smooth at  $r = 0$ , we obtain a radial eigenfunction of  $L^*$  for any  $\lambda$  with  $\text{Re } \lambda < d$ . This proves  $\{z \in \mathbb{C} : \text{Re } z < d\} \subset \sigma_p(L^*)$ , as desired.

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